How to Motivate Innovation: Subsidies or Prizes?*

Qiang Fu†
National University of Singapore

Jingfeng Lu‡
National University of Singapore

Yuanzhu Lu§
Central University of Finance and Economics

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Abstract

This paper investigates the optimal design of research contests. A "principal", who values an innovative technology, attempts to speed up the discovery. In order to minimize the expected amount of innovation time required, the principal decides how to allocate the fixed budget between a top-up prize (e.g. a procurement contract) and efficiency-enhancing subsidies (e.g. research grants) to competing R&D firms. Our paper shows that although both subsidies and prize incentives facilitate success, their functions differ subtly and the ability of one to substitute the other is limited. The main results are as follows. Firstly, the optimal contest preferentially subsidizes the ex ante less efficient firm. Secondly, more resources are devoted to research subsidies when the private benefit of the innovation to the successful innovator increases. Finally, more resources are allocated as subsidies when the innovation process involves more uncertainty.

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†Qiang Fu: Department of Business Policy, National Univeristy of Singapore, 1 Business Link, SINGAPORE, 117592; Tel: (65)65163775, Fax: (65)67795059, Email: bizfq@nus.edu.sg.

‡Jingfeng Lu: Department of Economics, National University of Singapore, 10 Kent Ridge Crescent, SINGAPORE, 119260; Tel: (65)65166026, Fax: (65)67752646, Email: ecsljf@nus.edu.sg.

§Yuanzhu Lu: China Economics and Management Academy, Central University of Finance and Economics, No 39 South College Road, CHINA, 100081; Tel: (8610)622288397, Fax: (8610)62288376, Email: yuanzhulu@cufe.edu.cn.

1 Introduction

R&D contests are frequently organized to promote the development of new technologies. As early as in 1714, the British Parliament offered a prize of £20,000 for a "practical and useful" means of determining longitude at sea (Che and Gale, 2003). In 1829, the directors of the Liverpool and Manchester Railway set aside a winner-take-all premium of £500 for the designer of the most improved locomotive engine for the first-ever passenger line (Day, 1971). The winning design propelled the world into the golden era of steam locomotion. Since these early days, the R&D contest mechanism has seen tremendous growth in its status as an efficient means of promoting innovation. The U.S. Department of Defense (DoD), for instance, has frequently sponsored design competitions to stimulate research into defense technology. One recent event was the 2005 DARPA (Defense Advanced Research Projects Agency) Grand Challenge, which is a 132.2 mile long race among autonomous robots in the Mojave Desert. The long list of prizes available today stands in testimony of the popularity and importance of R&D contests as a tool of innovation (see Konrad, 2007).

Besides prizes that reward winners, the sponsoring entity of a R&D contest may financially subsidize competing parties to provide additional incentives for innovative activities. The DoD, for instance, often heavily subsidizes firms that are shortlisted to participate in its design competitions (see Lichtenberg, 1988, and Che and Gale, 2003). In particular, it sets aside a significant portion of its R&D budget to be awarded to small business that engage in military research. As documented by Lichtenberg (1988), financial subsidies provided to contractors have substantially improved the productivity arising out of private military and reduced their marginal cost.

If properly administered, both a prize and a subsidy can provide competing parties with incentives with which to exert productive effort. It remains unclear, however, to what extent the two instruments could either substitute or complement each other functionally. This ambiguity poses a challenge to the design of a research contest, especially when the sponsoring entity is subject to fixed financial constraints. *How to choose between these two instruments to maximally foster certain innovation by a given amount of resource?* This paper addresses this question by studying the nature of these strategic instruments and the optimal design of a research contest.

In this paper, a scenario is considered in which two R&D firms engage in scientific activities in order to produce a given innovation (e.g., the longitude contest held in 1714) sooner than the other.[1] The firm that achieves this discovery first can patent the technology.

[1] Taylor (1995), Fullerton and McAfee (1999) and Che and Gale (2003) have demonstrated across contexts that it is optimal to shortlist only two contestants in such an R&D contests. In Appendix A, we demonstrate that the main insights of this paper are valid also for contests with three participants.

The winner enjoys a private benefit from this patent, including cash flow from royalties, procurement contracts from future buyers, and revenues from the commercialization of this technology.

A third party called the "principal" also benefits from this innovation. The principal can be a public agency that looks for a particular technology to fulfill its own goal. For instance, a country's Ministry of Health may demand an effective vaccine to rein in a deadly epidemic. Another example is that of the DoD, which actively seeks reliable combative robots to fight off snipers in Baghdad. The principal can also be a firm that searches for a technical solution that will facilitate the winning of greater market share. For instance, Apple Inc. has frequently outsourced its research and manufacturing tasks to capable high-tech firms. Alternatively, the principal can be a non-profit science foundation dedicated to inspiring scientific breakthroughs, such as resolving a major mathematical puzzle.

The principal attempts to speed up this discovery by optimally utilizing its limited financial resources. Subject to a fixed budget, the principal has the flexibility to either promise a top-up prize (e.g., an additional procurement contract) for the winner in order to step up the firms' research effort, or provide subsidies (e.g., research grants) to the firms in order to improve their efficiency.[2]

In this paper, rivaling firms compete to develop an innovation and the goal of the principal who stands to benefit from the successful development of the technology is to maximally reduce the length of the delivery cycle. We explicitly assume that the principal utilizes its resources solely to minimize the expected innovation time. As will be shown at a later point, minimizing the expected innovation time is equivalent to maximizing the present benefit that accrues to the principal.[3]

The expected innovation time is jointly determined by the outputs of participating firms. The technological output of a firm subsequently depends on its input (autonomous research effort) and its research capacity or efficiency (the quality of laboratory equipment and scientists). While a more generous prize purse encourages firms to step up their efforts and clearly contributes to a quicker discovery, the availability of greater research subsidies can also affect the innovation time in multiple ways. Such subsidies can work to speed up innovation directly by enhancing the firms' efficiencies and by amplifying the technological outputs the firms can produce given their inputs. On the other hand, the improved efficiency rendered by the research subsidy may alter the firms' incentive to exert research effort and the nature and magnitude of this indirect effect has yet to be identified in the literature. Given the

[2]The research grants can be used by the recipients to buy new instruments and hire more researchers.

[3]The contest considered in our scenario differs from the mechanisms of previous studies on research tournaments. Specifically, while Taylor (1995), Fullerton and McAfee (1999) and Che and Gale (2003) study the competition of firms on the basis of the quality of their output, firms in our scenario attempt to produce an innovation of given nature sooner than the others.

fixed budget of the principal, providing research subsidies dilutes the resource that would otherwise be put into a more generous prize purse and may blur the trade-off between the two options.

Our model provides a framework for the analysis on the optimal design of research contests in which financial resources are limited. Although both subsidies and prize incentives facilitate success, their functions differ subtly and the ability of one to substitute the other is limited. In particular, the optimal budget allocation depends critically on the subtle interactions among three main factors: the existing value of the patent (private benefit) to the successful innovator, the technological nature of this innovation and the initial technological endowments (research capacity) of the firms. We highlight our main observations below.

First, we find that the incidence and amount of research subsidies are positively related to the amount of private benefit the successful innovator can receive from the patent. Innovations vary in the amount of private benefit that can be gained by the winner. For instance, an applied research project could potentially lead to considerable commercial value, while basic research typically does not. Similarly, a civil technology can be widely licensed to reap immediate rewards, while military technology may not. The potentially lower level of private benefit could thus propel the principal to compensate for this missing incentive by enriching the value of the prize purse. By way of contrast, when the patent itself provides substantial return, an additional prize purse would have a lesser effect on firms' incentives and more resources might consequently be allocated as research subsidies.[4]

Second, we predict that when the development of the targeted technology involves greater uncertainty, the optimum would entail a lower level of subsidy and a more generous prize purse. In the current scenario, the amount of uncertainty involved in the development process is specifically measured by the elasticity of the hazard rate (the conditional likelihood of discovery) to a firm's effort supply. That is, the higher the elasticity, the more a given amount of incremental effort can contribute to success, and the lesser the uncertainty or difficulty involved. Innovation projects differ in this aspect. For example, additional research inputs could significantly expedite the development of a diet drug, while it would do less for the development of an effective HIV vaccine. This result provides important insights into the design of incentive mechanisms that facilitate innovation: prize incentives will sufficiently motivate targeted research only if the project of interest involves a moderate level of difficulty, while subsidies provide stronger motives when the development process is subject to more uncertainty and competing parties hold weak prospects of success.

Third, we show that the optimally designed contest preferentially subsidizes the initially less efficient firm. Firms may differ in their initial capabilities due to heterogeneity in terms

[4]Enhanced research capacities can thus induce contestants to exert a greater amount of research effort. This has been empirically evidenced by Lach (2002), among others.

of their physical capacity, human capital and knowledge stock.[5] A subsidy provides a catalyst for success in so far as it amplifies the output of any given effort input. However, its impact on the equilibrium effort supply is mixed. On the one hand, it could encourage the recipient to work harder, as a subsidy is qualitatively equivalent to a reduction in the marginal cost of effort. On the other hand, it tilts the balance of the tournament field, and indefinitely mediates the competing firms' incentive to supply effort. A subsidy could maximally shorten the delivery cycle only if it levels the tournament field; while it provides a negative incentive (i.e., reduces equilibrium effort supply) when it exacerbates an initial asymmetry.

This finding sheds light on the SBIR (Small Business Innovation Research) program that the U.S. Government created in 1983, in which the DoD was a participating agency. The program dedicates funds to assist small businesses in their innovative activities and enables them "to undertake and to obtain the benefits of research and development in order to maintain and strengthen the competitive free enterprise system and the national economy".[6] Our analysis demonstrates the efficiency gains that may be expect from such a preferential funding system, over and above its normative value.

A novel feature of our paper is that it explicitly includes financial subsidies in the portfolio of strategic instruments.[7] A number of studies have concerned themselves with the optimal allocation of prizes of differing ranks for innovation related contests and for other types of contests. For instance, Denicolò and Franzoni (2007) investigated the optimal use of the winner-take-all principle in innovation races conducted under different market conditions.[8] However, an important dimension of the analysis on optimal contest design has yet to be systematically investigated, specifically, that the organizer may allocate her budget to improving participants' capabilities, as well as to enriching the winner's purse.[9] Our paper contributes to the literature on the optimal design of research contest by adding to the literature in this aspect.

This paper is also closely linked to the literature on optimal handicapping. Conventional wisdom teaches that a more level playing field creates more competition. A handful of papers

[5] The heterogeneity of firms can also be attributed to their financial liquidity . Financial constraints that restricts a firm's action space are not explicitly included in this paper. However, the effect of a financial constraint on effort supply can technically be interpreted as a low efficiency parameter in our model. More details will be revealed by the model, and these will be discussed in Section 5.

[6] Source: *Defense Small Business Innovation Research Program Review Summary*, ExpectMore.gov.

[7] While Fu and Lu (2008b) and Moldovanu, Sela and Shi (2008) allow contest organizer to impose finanical punishment or to charge entry fees, the current paper instead assumes limited liability and allows only nonnegative resource transfers to competing parties.

[8] Other studies include that by Moldovanu and Sela (2001), Rosen, (1986), and Fu and Lu (2009b).

[9] Contestants may also be motivated to make pre-contest investments of their own to improve their competitive standing, and this scenario has recently been analyzed by Kräkel (2004), Münster (2007) and Fu and Lu (2009a), among others.

have analytically implemented this logic in different contexts. For instance, Che and Gale (2003) show that in a research contest in which firms compete based on the quality of the technology, imposing a bidding cap encourages both contestants to step up their effort supply. Unlike Che and Gale (2003), however, this paper studies research contests where firms race toward a specifically defined innovation, and the mechanism used seeks to positively assist the weaker instead of handicapping the stronger.[10] Both approaches aim to balance the competition, and this paper complements Che and Gale's (2003) in this regard.

For the sake of expositional convenience and tractability, we use a stylized setting of two firms for the current analysis. However, our findings are largely robust when changes are made to the number of firms, when there is a time discount factor, and when the firms' are constrained by their own limited resources. These extensions will be discussed in Section 5 and Appendix A.

The remainder of this paper is structured as follows. Section 2 presents the setup of the model. Section 3 executes the formal analysis on optimal contest design. Section 4 discusses the results. Section 5 concludes the paper, and reviews the robustness of our results in more extensive settings.

2 The Model

Two R&D firms, indexed by $i = 1, 2$, are engaged in a race towards an innovative technology. The firm that succeeds first can patent this technology and secures a private benefit $V_0 \geq 0$. A third party (e.g. a public agency like DoD) benefits from this innovation, and hence is eager to foster this technology by using a limited budget of M. This party is generically named a "principal". The principal has full discretion to divide her budget M into two parts: direct subsidies to firms (S_1 and S_2) and a top-up winning prize (Γ_0), e.g. a procurement contract, with $S_1, S_2, \Gamma_0 \geq 0$.[11] Specifically, we assume that the principal attempts to elicit the innovative effort supplied by the two competing R&D firms to minimize the expected innovation time. This objective function of the principal will be further elaborated upon in Section 2.3.

[10] In an earlier study Che and Gale (1998) studied the effect of a contribution cap on influence politics. They showed that such a cap serves to intensify the competition as it diminishes the advantage of the stronger bidder. Similarly, Clark and Riis (2000) and Fu (2006) have shown that in an all-pay auction with heterogenous players, total effort may increase when the weak contestant's bid is properly scaled up.

[11] It is assumed throughout this paper that the R&D firms are subjected to limited liability, which thus requires that the monetary transfer S_i from the principal be non-negative.

2.1 The Timing of Moves

We consider a two-stage game. The sequence of moves in this game is the following. First, the principal announces the rule of the contest, which is represented by the profile (S_1, S_2, Γ_0). She strategically sets the amounts of research subsidies and the top-up prize to minimize the expected innovation time. Second, observing the profile (S_1, S_2, Γ_0) set by the principal, firms simultaneously and independently commit to their R&D outlays x_1 and x_2 in order to maximize their expected payoffs.

2.2 The Race between Firms

Having observed the contest rule (S_1, S_2, Γ_0) announced by the principal, each firm i invests an R&D effort of x_i on this research project in order to achieve a quicker discovery. The first successful innovator receives a total winner's purse of $V_0 + \Gamma_0$ as its reward, while the other firm gets nothing. We adopt the framework of Dasgupta and Stiglitz (1980) to model this R&D race.[12,13] The actual time t_i for firm i to accomplish this task is a random variable that follows a Weibull (minimum) distribution. To put it formally, given x_i, the probability that firm i successfully innovates before time t is given by

$$F_i(t|x_i) = 1 - e^{-h_i(x_i)t},\ t \geq 0,\ i = 1, 2, \tag{1}$$

where $h_i(x_i)$ is firm i's hazard rate of success, i.e., $h_i(x_i)\Delta t$ measures firm i's conditional probability of making the discovery between time t and time $t + \Delta t$, provided that the discovery has not been achieved before time t. It is assumed that $h_i(x_i)$ is strictly increasing with effort x_i, and is concave in its argument. Clearly, $h_i(x_i)$ measures the technical output of firm i's innovation activities. Specifically, it is assumed, for analytical tractability, that the hazard rate $h_i(x_i)$ takes the functional form $h_i(x_i) = k_i x_i^r$, with $k_i > 0$ and $r \in (0, 1]$. The parameter r, which measures the elasticity of the hazard rate to additional effort, is common to both firms. Determined by the technological nature of this innovation, it indicates the effectiveness of the R&D efforts in this particular innovation activity.[14] A smaller r depicts an

[12] This approach assumes that each competing firm commits to a one-shot lump-sum R&D outlay that determines the time distribution of its success. Denicolò (2000) and Baye and Hoppe (2003), among many others, have followed this approach in modelling innovation races. Kaplan, Luski and Wettsein (2003) suggest an approach that is analogous to an all-pay auction but also requires lump-sum effort. Another popular approach has been proposed by Lee and Wilde (1980) that requires each competing firm to commit to its R&D investment rate. Please refer to Reinganum (1989) for a thorough discussion on these modelling approaches. A recent application of Lee and Wilde (1980)'s model can be seen in Etro (2004).

[13] As pointed out by Baye and Hoppe (2003) and Fu and Lu (2008a), this type of patent race model is of stochastic equivalence to a Tullock contest.

[14] See Fu and Lu (2008a) for further interpretations of parameter r. Fu and Lu (2008a) have shown that a patent race model is equivalent to a Tullock contest with impact functions of $h_i(\cdot)$.

innovation that involves more uncertainty. Parameter k_i measures a firm i's research capacity (e.g., quality of laboratory instruments and scientists), or the efficiency or capability of this firm. The two firms could differ ex ante in their initial endowments, and a subsidy from the principal directly improves the recipient firm's capacity. It is assumed that $k_i \equiv \theta_i + S_i$, where θ_i indicates the firm's initial capacity, while S_i denotes the research subsidy the firm receives from the principal. As a result, a firm i's hazard rate of success is given by

$$h_i(x_i) = (\theta_i + S_i)x_i^r. \tag{2}$$

This setting thus intuitively captures the notion that a research subsidy amplifies a firm's R&D output. For example, the firm could spend the subsidy upgrading laboratory equipment and hiring additional scientists. Additional physical or human capital stock boosts the firm's productivity, as it allows the firm to conduct more parallel experiments. Without loss of generality, it is assumed that $\theta_1 \geq \theta_2$, which indicates that Firm 1 is ex ante more efficient than Firm 2.

It is assumed that R&D effort incurs a unitary marginal cost to each firm, i.e., $C_i(x_i) = x_i$.[15] Hence, a firm i's expected payoff is given by

$$\pi_i(x_i, x_j) = \Pr(t_i < t_j | x_i, x_j) V - x_i,\ i = 1, 2,$$

where $V \equiv V_0 + \Gamma_0$ is the total reward received by the winner, which includes the private benefit from the patent V_0 and the value of the top-up prize Γ_0 awarded by the principal.

2.3 The Principal's Optimization Problem

From the viewpoint of the principal, for a given effort profile (x_1, x_2), the innovation time has a cumulative distribution function

$$\begin{aligned} F(t|x_1, x_2) &= 1 - (1 - F_1(t|x_1))(1 - F_2(t|x_2)) \\ &= 1 - e^{-[h_1(x_1)+h_2(x_2)]t},\ \ t \geq 0. \end{aligned} \tag{3}$$

The expected innovation time is then given by

$$E(t|x_1, x_2) = \frac{1}{h_1(x_1) + h_2(x_2)}. \tag{4}$$

The principal, who attempts to minimize (4), has to determine the optimal research subsidy profile $(S_1,\ S_2)$ that subsidizes the competing firms, and a top-up prize Γ_0 that

[15] Firms may differ in terms of the costs they spent on R&D. However, we do not explicitly include asymmetry in this aspect. The asymmetry in terms of productivity can be reflected by the difference in efficiency parameter θ_i, as a higher marginal product of effort implies that a given amount of technological output requires a lesser cost.

rewards the winner. She is subject to the following budget constraint:

$$S_1 + S_2 + \ \Gamma_0 \leq M,\ S_1 \geq 0,\ S_2 \geq 0,\ \ \Gamma_0 \geq 0. \tag{5}$$

An Alternative Derivation of the Principal's Objective

It has explicitly been assumed that the principal solely minimizes the expected innovation time. This objective can be alternatively motivated by the following derivation. Assume that the principal receives a benefit B for each unit of time from consuming the innovation, and that it has a time discount rate of ρ. Let the principal attempt to maximize the present benefit that can be gained by the successful delivery of the innovation. We claim the following.

Theorem 1 *Maximizing present benefit the principal can receive from the innovation is equivalent to minimizing expected innovation time.*

Proof. When maximizing her present benefit, the principal has the following objective function $U \equiv U(h_1(x_1), h_2(x_2), \rho)$:

$$\begin{aligned} & U(h_1(x_1), h_2(x_2), \rho) \\ = & \int_0^\infty \left(\int_t^\infty e^{-\rho s} B ds \right) (h_1(x_1) + h_2(x_2)) e^{-(h_1(x_1)+h_2(x_2))t} dt \\ = & \frac{B}{\rho}[1 - \frac{\rho}{h_1(x_1) + h_2(x_2) + \rho}]. \end{aligned} \tag{6}$$

Thus, the principal maximizing (6) in fact minimizes the expected innovation time as given by (4).

Q.E.D. ■

As we can observe from (6) and (4), both objectives (maximizing U and minimizing $E(t)$) are equivalent to maximizing the aggregate technological output $[h_1(x_1) + h_2(x_2)]$.

For the sake of analytical tractability, it is assumed that the firms do not discount future payoffs. This setting, however, corresponds to a situation in which early innovation is more appealing to the principal than to the innovators.[16] For instance, discovering an effective vaccine timely could be more important to a public health agency as a sponsor who badly needs the vaccine to tackle an epidemic disease than to the private innovators who might only care about their own benefits.

[16] Please refer to Section 5.2 for a more detailed discussion of this.

3 The Analysis

We solve for the subgame perfect Nash equilibrium of the two-stage game via backward induction. Each firm's equilibrium effort outlay in every given subgame is first characterized. We then search for the optimally designed contest $(S_1^*, S_2^*, \Gamma_0^*)$ announced by the principal, when it anticipates the equilibrium responses of the competing firms $(x_1(S_1^*, S_2^*, \Gamma_0^*), x_2(S_1^*, S_2^*, \Gamma_0^*))$.

3.1 Equilibrium R&D Effort

A firm wins if it realizes the desired discovery sooner than the other. The winning probability of a firm i is then given by

$$\begin{aligned} \Pr\left(t_i < t_j | x_i, x_j\right) &= \int_0^\infty \left(\int_0^{t_j} F_i'\left(t_i | x_i\right) dt_i \right) F_j'\left(t_j | x_j\right) dt_j \\ &= \frac{h_i(x_i)}{h_i(x_i) + h_j(x_j)}. \end{aligned}$$

Recall that the hazard rate $h_i(x_i) = k_i x_i^r = (\theta_i + S_i) x_i^r$. Firm i's expected payoff is thus rewritten as

$$\begin{aligned} \pi_i(x_i, x_j) &= \Pr\left(t_i < t_j | x_i, x_j\right) V - x_i \\ &= \frac{(\theta_i + S_i) x_i^r}{(\theta_i + S_i)\, x_i^r + (\theta_j + S_j)\, x_j^r} \left(V_0 + \Gamma_0\right) - x_i,\ i = 1, 2. \end{aligned}$$

The first-order condition for the firm i's expected payoff maximization problem is given by

$$\frac{\partial \pi_i}{\partial x_i} = \frac{r x_i^{r-1} x_j^r (\theta_i + S_i)\,(\theta_j + S_j)}{\left[(\theta_i + S_i)\, x_i^r + (\theta_j + S_j)\, x_j^r\right]^2} \left(V_0 + \Gamma_0\right) - 1 = 0,\ i = 1, 2.$$

The equilibrium effort for a given allocation profile (S_1, S_2, Γ_0) is thus obtained as follows

$$x_1^* = x_2^* = x^* = \frac{r(\theta_1 + S_1)\,(\theta_2 + S_2)}{\left[(\theta_1 + S_1) + (\theta_2 + S_2)\right]^2} \left(V_0 + \Gamma_0\right). \qquad (7)$$

Notably, the two firms invest the same amount of R&D effort in the equilibrium, regardless of the levels of their ex ante research capacities.[17] A more efficient firm does not have to exert more effort than its rival. The asymmetry between the firms is reflected by the differing levels of resulted equilibrium output $k_i x_i^r$, which guarantees that the more efficient firm stands a better chance of winning the race.[18]

[17] However, as we will show in Appendix A, firms may or may not exert the same amount of effort in equilibrium when three or more firms participate in the competition.

[18] It can be shown that each firm's expected payoff is positive, so it is always incentive compatible for the two firms to participate in the contest regardless of the principal's choices.

3.2 Optimal Budget Allocation

Having obtained the contestants' equilibrium effort outlays in any given contest (S_1, S_2, Γ_0), the principal's optimal budget allocation problem is now probed. The principal's objective is to minimize the expected innovation time. Given the equilibrium effort function (7), the expected innovation time is thus written as

$$E(t) = \frac{1}{[(\theta_1 + S_1) + (\theta_2 + S_2)]x^{*r}} = \frac{[(\theta_1 + S_1) + (\theta_2 + S_2)]^{2r-1}}{[r(\theta_1 + S_1)(\theta_2 + S_2)(V_0 + \Gamma_0)]^r}. \quad (8)$$

The principal is to set the optimal bundle $(S_1^*, S_2^*, \Gamma_0^*)$ to minimize (8), subject to constraints (5). The principal allocates the resources among the three elements by comparing their marginal impacts on $E(t)$, which are given by

$$\frac{\partial E(t)}{\partial S_1} = E(t)\left[\frac{2r-1}{\theta_1 + \theta_2 + S_1 + S_2} - \frac{r}{\theta_1 + S_1}\right], \quad (9)$$

$$\frac{\partial E(t)}{\partial S_2} = E(t)\left[\frac{2r-1}{\theta_1 + \theta_2 + S_1 + S_2} - \frac{r}{\theta_2 + S_2}\right], \quad (10)$$

$$\text{and } \frac{\partial E(t)}{\partial \Gamma_0} = E(t)\frac{-r}{V_0 + \Gamma_0}. \quad (11)$$

Before formally solving for the optimal budget allocation plan $(S_1^*, S_2^*, \Gamma_0^*)$, we first examine the role played by these strategic instruments.

Lemma 1 *Let ς denote any of the three instruments (S_1, S_2, Γ_0). $\frac{\partial E(t)}{\partial \varsigma} < 0$ and $\frac{\partial^2 E(t)}{\partial \varsigma^2} > 0$, regardless of the existing allocation (S_1, S_2, Γ_0).*

Proof. See Appendix B. ■

Lemma 1 manifests the constantly positive effects of the three strategic instruments (S_1, S_2, Γ_0) on quicker success. Regardless of the existing allocation (S_1, S_2, Γ_0), a more generous prize purse or additional subsidy to either recipient always helps reduce $E(t)$. However, the marginal impact of an instrument strictly decreases as the amount of resources allocated to the instrument increases. As a result of Lemma 1, the optimal allocation profile requires the budget of the principal to be binding.

Obviously, a more generous prize purse encourages both firms to step up their effort, which unambiguously leads to a speedier success. However, an additional subsidy plays a subtler role. A subsidy to a firm affects the expected innovation through two avenues. On the one hand, an additional subsidy to a firm (directly) amplifies the firm's technological output for any given profile of effort outlays. Recall that the expected innovation time $E(t)$ is given by $E(t) = \frac{1}{[(\theta_1+S_1)+(\theta_2+S_2)]x^{*r}}$. This direct effect is apparently negative (i.e., it reduces the expected innovation time). Its size does not depend upon the identity of the recipient for any given x^*. On the other hand, an additional subsidy indirectly affects the innovation time as it varies the equilibrium effort x^*. The direction of the indirect effect, however, critically depends on the identity of the recipient.

Take the first order partial derivative of x^* with respect to S_i and we obtain

$$\frac{\partial x^*}{\partial S_i} = \frac{r\left(V_0 + \Gamma_0\right)\left(\theta_j + S_j\right)}{\left[\left(\theta_i + S_i\right) + \left(\theta_j + S_j\right)\right]^3} \cdot \left[\left(\theta_j + S_j\right) - \left(\theta_i + S_i\right)\right]. \tag{12}$$

The sign of $\frac{\partial x^*}{\partial S_i}$ thus depends on $(\theta_j + S_j) - (\theta_i + S_i)$. It reveals the nature of this indirect effect on firms' competitive incentives: when an additional subsidy is provided to a firm, both of firms step up (down) their equilibrium effort x^* if and only if the recipient is less (more) efficient than the other, i.e., $\frac{\partial x^*}{\partial S_i} \gtreqless 0$ if and only if $(\theta_i + S_i) \lesseqgtr (\theta_j + S_j)$. A subsidy spurs on additional competition when it levels the playing field, while stifling the competition when it exacerbates existing imbalance. As the amount of time for innovation strictly decreases in the effort exerted by each firm, the implication of these observations becomes clear. We then obtain Lemma 2.

Lemma 2 *An optimal budget allocation must satisfy the following conditions:*

(a) Firm 1 remains at least as efficient as Firm 2, i.e., $\theta_1 + S_1^* \geq \theta_2 + S_2^*$;

(b) If Firm 1 receives a positive subsidy, then Firm 2 must be as strong as Firm 1, i.e., $\theta_1 + S_1^* = \theta_2 + S_2^*$ *if* $S_1^* > 0$;

(c) If the budget allocation allows firm 1 to remain strictly more efficient than Firm 2, i.e., $\theta_1 + S_1^* > \theta_2 + S_2^*$, *then it must receive no subsidy, i.e.,* $S_1^* = 0$.

Proof. See Appendix B. ■

Although the indirect effect never dominates the constantly positive direct effect, in order to maximally reduce the amount of time that firms require to achieve the innovation with a limited amount of resources, the optimal allocation bundle would preferentially subsidize the ex ante weaker firm. However, the optimum never allows this firm to ex post leapfrog the other.

Hence, in the subsequent analysis, two possible cases are considered: (1) $\theta_1 - \theta_2 > M$ and (2) $\theta_1 - \theta_2 \leq M$. In the former case, severe capacity asymmetry exists across the firms, and the resource available to the principal does not suffice to fill in the gap. Consequently, Firm 1 would remain strictly more efficient than Firm 2 regardless of the allocation plan. In the latter case, the initial asymmetry between the two firms is relatively mild and the principal could fully balance the competition using its budget, although a fully symmetric race may not be optimal.

3.2.1 The Case of Severe Asymmetry ($\theta_1 - \theta_2 > M$)

The following preliminary result on the distribution of a research grant between the two firms is first established.

Lemma 3 *When $\theta_1 - \theta_2 > M$, the optimally designed contest does not subsidize Firm 1, i.e., $S_1^* = 0$.*

Lemma 3 directly follows from Lemma 2(c). When the funds available to the principal are insufficient to fully balance the playing field, any subsidy to Firm 1 provides only a negative incentive to the firms as it further upsets the balance. Hence, three possible optimal allocation plans could result: (1) $S_2^* = M$, and $\Gamma_0^* = 0$, (2) S_2^*, $\Gamma_0^* > 0$, $S_2^* + \Gamma_0^* = M$ and (3) $S_2^* = 0$, $\Gamma_0^* = M$.

In view of the fact that Firm 1 never receives any subsidy in this case, (8) leads to

$$E(t) = \frac{[\theta_1 + (\theta_2 + S_2)]^{2r-1}}{[r\theta_1 (\theta_2 + S_2)(V_0 + \Gamma_0)]^r}. \tag{13}$$

The principal minimizes (13) subject to the constraint $S_2 + \ \Gamma_0 = M$, $S_2 \geq 0$, $\ \Gamma_0 \geq 0$. (13) leads to

$$\begin{aligned} \frac{\partial E(t)}{\partial S_2} &= E(t) \cdot [\frac{2r-1}{\theta_1 + (\theta_2 + S_2)} - \frac{r}{\theta_2 + S_2}]; &(14) \\ \text{and } \frac{\partial E(t)}{\partial \Gamma_0} &= E(t) \cdot \frac{-r}{V_0 + \Gamma_0}. &(15) \end{aligned}$$

The optimal budget allocation between Γ_0 and S_2 is searched for by conducting the following thought experiment. An arbitrary allocation plan $(S_2, \Gamma_0 = M - S_2)$ is fixed. The principal may have to reallocate the resource between S_2 and Γ_0 to achieve the optimum. The direction of desirable reallocation thus completely depends on a comparison between the impacts of the two instruments. In other words, when $\left|\frac{2r-1}{\theta_1+(\theta_2+S_2)} - \frac{r}{\theta_2+S_2}\right| > \left|\frac{-r}{V_0+M-S_2}\right|$, the resource must be directed away from the prize but towards S_2, while otherwise it goes the other way.

The trade-off between the subsidy and top-up prize is illustrated in Figure 1. The values for the parameters are as follows: $\theta_1 = 4$, $\theta_2 = 1$, $r = 1$, $V_0 = 1$, $M = 2$.

The downward-sloping solid curve plots $|\frac{\partial E(t)}{\partial S_2}|$, while the upward-sloping dotted curve plots $|\frac{\partial E(t)}{\partial \Gamma_0}|$. As the optimum requires a binding budget constraint of $S_2 + \Gamma_0 = M$, $\frac{\partial E(t)}{\partial S_2}$ and $\frac{\partial E(t)}{\partial \Gamma_0}$ are simply defined as functions of S_2 instead of both S_2 and Γ_0. As shown in Figure 1, when S_2 increases, its marginal impact on $E(t)$ continues to decline, while it complementarily steps up the magnitude of $\frac{\partial E(t)}{\partial \Gamma_0}$. These observations tremendously facilitate the derivation of the following results.

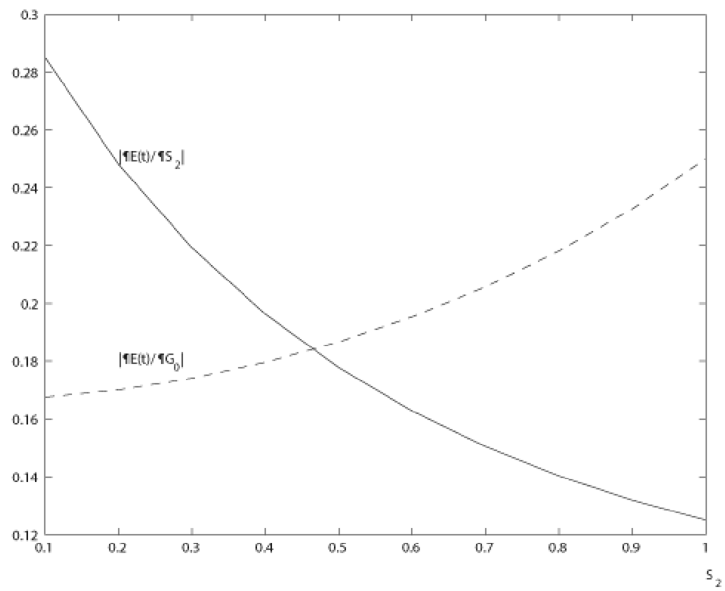

Figure 1: The Impacts of S_2 and Γ_0 on $E(t)$

Proposition 1 *When $\theta_1 - \theta_2 > M$, in the unique subgame perfect equilibrium, the principal*

(a) exhausts the entire budget to subsidize Firm 2, i.e., $S_2^* = M$, if and only if $V_0 \geq \frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}$;

(b) allocates the entire budget to the prize, i.e., $\Gamma_0^* = M$, if and only if $V_0 + M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$;

(c) splits its budget between the prize and the subsidy to Firm 2, if and only if

$$V_0 < \frac{r(\theta_1 + \theta_2 + M)(\theta_2 + M)}{r\theta_1 + (1-r)(\theta_2 + M)} \text{ and } V_0 + M > \frac{r\theta_2(\theta_1 + \theta_2)}{r\theta_1 + (1-r)\theta_2}. \tag{16}$$

In this case, the optimal $S_2^* \in (0, M)$ is the unique solution of

$$\frac{r}{\theta_2 + S_2} - \frac{2r-1}{\theta_1 + (\theta_2 + S_2)} = \frac{r}{V_0 + M - S_2}. \tag{17}$$

Proof. (a) Note $\frac{r}{\theta_2+S_2} - \frac{2r-1}{\theta_1+(\theta_2+S_2)}$ strictly decreases with S_2, and $\frac{r}{V_0+M-S_2}$ strictly increases with S_2. Thus it is optimal to have $S_2 = M$ if and only if $\frac{r}{\theta_2+S_2} - \frac{2r-1}{\theta_1+(\theta_2+S_2)} \geq \frac{r}{V_0+M-S_2}$ when $S_2 = M$. This requires

$$\frac{r}{\theta_2 + M} - \frac{2r-1}{\theta_1 + (\theta_2 + M)} \geq \frac{r}{V_0} \Leftrightarrow \frac{r[\theta_1 + (\theta_2 + M)]\,(\theta_2 + M)}{r\theta_1 + (1-r)\,(\theta_2 + M)} \leq V_0.$$

(b) By the same argument as the proof for (a), it is optimal to allocate the entire budget to Γ_0 if and only if $\frac{r}{\theta_2+S_2} - \frac{2r-1}{\theta_1+(\theta_2+S_2)} \leq \frac{r}{V_0+M-S_2}$ when $S_2 = 0$, i.e. $\Gamma_0 = M$. This requires

$$\frac{r}{\theta_2} - \frac{2r-1}{\theta_1 + \theta_2} \leq \frac{r}{V_0 + M} \Leftrightarrow V_0 + M \leq \frac{r\theta_2(\theta_1 + \theta_2)}{r\theta_1 + (1-r)\theta_2}.$$

However, such an equilibrium of (b) may not exist at all. The above condition can hold only if $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} > M$.

(c) The results of (a) and (b) directly imply (c). When an interior solution ends up, the equilibrium would require (17) hold. Note that the left hand side of (17) strictly decreases with S_2, and the right side of (17) strictly increases with S_2. By intermediate value theorem, a unique solution $S_2^* \in (0, M)$ of (17) must exist if the conditions of Proposition 1(c) hold.

Q.E.D. ■

Proposition 1 states that the optimal budget allocation profile crucially depends on the private benefit V_0. A subsidy therefore gains more appeal. More specifically, the principal should expend more resources subsidizing the ex ante weaker firm when the firms expect ample rewards from the patent, while it should expend more resources augmenting the prize purse when the patent value is insufficient.

When the amount of private benefit falls within a medium range, an interior solution results and gives rise to both a positive subsidy and a positive top-up prize. To explicitly solve for the optimal subsidy S_2^*. Rearrange (17) and the following quadratic equation is obtained:

$$S_2^{*2} + A_1 S_2^* - A_2 = 0, \tag{18}$$

where $A_1 \equiv r(2\theta_1+\theta_2)+(1-r)(V_0+M)+\theta_2$ and $A_2 \equiv (r\theta_1 + (1-r)\theta_2)(V_0+M) - r\theta_2(\theta_1+\theta_2)$. The following corollary thus follows.

Corollary 1 *When* $V_0 < \frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}$ *and* $V_0 + M > \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$*, the principal allocates an amount* $S_2^* = \frac{\sqrt{A_1^2+4A_2}-A_1}{2}$ *to subsidize the weaker firm.*

Proof. See Appendix B. ■

3.2.2 The Case of Mild Asymmetry ($\theta_1 - \theta_2 \in [0, M]$)

The case that involves mildly asymmetric firms is now considered. When $\theta_1 - \theta_2 \in [0, M]$, the principal has sufficient funds to fully counterbalance the asymmetry by preferentially subsidizing the weaker firm. However, a fully balanced playing field may not necessarily emerge in the optimum, as a more generous prize purse stimulates effort supply as well.

Returning to the thought experiment conducted for the case of severe asymmetry, an arbitrary allocation plan is fixed and desirable reallocation is searched out. There are altogether two possible (initial) scenarios. In the first, a positive subsidy is provided to Firm 1, i.e., $S_1 > 0$. Lemma 2(b) thus requires $S_2 = S_1 + (\theta_1 - \theta_2)$ and $\Gamma_0 = M - (\theta_1 - \theta_2) - 2S_1$. That is, the principal fully offsets the initial imbalance. Note that the limiting situation $\Gamma_0 = 0$ is not excluded, while $(S_1 = 0, S_2 = \theta_1 - \theta_2)$ depicts the other limiting situation when $S_1 \to 0^+$. In the second scenario, no subsidy is provided to Firm 1, i.e., $S_1 = 0$. Lemma 1 thus implies that $S_2 \in [0, \theta_1 - \theta_2]$, where the asymmetry between firms continues to exist. Again, the limiting situation $\Gamma_0 = M$ is not excluded.

Assuming that the first scenario currently prevails, the optimum requires $S_2 = S_1 + (\theta_1 - \theta_2)$ and $\Gamma_0 = M - (\theta_1 - \theta_2) - 2S_1$. For all allocations that satisfy these conditions, the first order partial derivatives of (8) are given by

$$\frac{\partial E(t)}{\partial S_1} = \frac{\partial E(t)}{\partial S_2} = E(t) \cdot \frac{-1}{2(\theta_1 + S_1)}, \tag{19}$$

$$\text{and } \frac{\partial E(t)}{\partial \Gamma_0} = E(t) \cdot \frac{-r}{V_0 + M - (\theta_1 - \theta_2) - 2S_1}. \tag{20}$$

Again, the direction of a desirable reallocation depends on the magnitude of the RHS of (19) and (20). An additional subsidy is desirable if and only if $\frac{1}{2(\theta_1+S_1)} > \frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1}$. By way of contrast, an additional prize purse is preferred if and only if $\frac{1}{2(\theta_1+S_1)} < \frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1}$, and a desirable reallocation requires that S_1 and S_2 be reduced by an equal amount until the resource that subsidizes Firm 1 is completely taken away. When Firm 1 receives zero subsidy, the second scenario thus emerges, and the first order derivatives of $E(t)$ boil down to (14) and (15). Repeating the practice in Section 3.2.1 leads to the following results regarding the optimal allocation plan.

Proposition 2 *When $\theta_1 - \theta_2 \leq M$, in the unique subgame perfect equilibrium, the principal*

(a) allocates the entire budget to subsidize the two firms, i.e., $S_1^* = \frac{M-(\theta_1-\theta_2)}{2}, S_2^* = \frac{M+(\theta_1-\theta_2)}{2}$ and $\Gamma_0^* = 0$ if and only if $V_0 \geq r(\theta_1 + \theta_2 + M)$;

(b) subsidizes both firms and creates a positive prize, i.e., $S_1^* = S_2^* - (\theta_1 - \theta_2) > 0$ and $\Gamma_0^* > 0$ if and only if $V_0 < r(\theta_1 + \theta_2 + M)$ and $V_0 + M > 2r\theta_1 + (\theta_1 - \theta_2)$;

(c) allocates the entire budget to the prize, i.e., $\Gamma_0^* = M$, if and only if $V_0 + M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$;

(d) splits the budget between the subsidy S_2 to the weaker firm and the prize, i.e., $S_1^* = 0, S_2^* \in (0, \theta_1 - \theta_2]$, and $\Gamma_0^* = M - S_2^* \geq M - (\theta_1 - \theta_2)$, if and only if $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} < V_0 + M < 2r\theta_1 + (\theta_1 - \theta_2)$.

Proof. (a) In scenario one, we have $\frac{1}{2(\theta_1+S_1)}$ strictly decreases with S_1 while $\frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1}$ strictly increases with S_1. An optimum that involves zero Γ_0 would emerge if and only if $\frac{1}{2(\theta_1+S_1)} \geq \frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1}$ when $S_1 = \frac{M-(\theta_1-\theta_2)}{2}$ (and $S_2 = \frac{M+(\theta_1-\theta_2)}{2}$). This condition can be written as

$$\frac{1}{\theta_1 + \theta_2 + M} \geq \frac{r}{V_0}, \tag{21}$$

which is equivalent to $V_0 \geq r(\theta_1 + \theta_2 + M)$.

Note that at allocation $(S_1 = 0, S_2 = \theta_1 - \theta_2)$, the comparison between (19) and (20) for scenario one is consistent with that between (14) and (15) for scenario two as the relevant derivatives take the same values (due to the continuity). Note that as long as the comparison between (14) and (15) for $(S_1 = 0, S_2 = \theta_1 - \theta_2)$ favors more research subsidies, then all allocations of scenario two are dominated by $(S_1 = 0, S_2 = \theta_1 - \theta_2)$.

The above arguments show that (21) is a sufficient condition for $\Gamma_0 = 0$. The necessity of this condition is clear as if it does not hold then deviating slightly from $\Gamma_0 = 0$ by creating positive top-up prize reduces the expected innovation time.

(b) Based on the arguments of case (a), to have the organizer subsidize both firms and create positive prizes, the necessary and sufficient condition is $\frac{1}{2(\theta_1+S_1)} = \frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1}$ for some $S_1 \in (0, \frac{M-(\theta_1-\theta_2)}{2})$ (and $S_2 = S_1 + (\theta_1 - \theta_2)$). This is equivalent to $\frac{1}{2(\theta_1+S_1)} < \frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1}$ when $S_1 = \frac{M-(\theta_1-\theta_2)}{2}$ (and $S_2 = \frac{M+(\theta_1-\theta_2)}{2}$) and $\frac{1}{2(\theta_1+S_1)} > \frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1}$ when $S_1 = 0$ (and $S_2 = \theta_1 - \theta_2$). Condition $\frac{1}{2(\theta_1+S_1)} < \frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1}$ when $S_1 = \frac{M-(\theta_1-\theta_2)}{2}$ (and $S_2 = \frac{M+(\theta_1-\theta_2)}{2}$) means deviating slightly from $\Gamma_0 = 0$ by creating some top-up prize reduces the expected innovation time. It can be written as $V_0 < r(\theta_1 + \theta_2 + M)$. The condition $\frac{1}{2(\theta_1+S_1)} > \frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1}$ when $S_1 = 0$ (and $S_2 = \theta_1 - \theta_2$) means that additional prize reduces the expected innovation time. The latter condition is satisfied if $V_0 + M > 2r\theta_1 + (\theta_1 - \theta_2)$. If $2r\theta_1 + (\theta_1 - \theta_2) \leq M$, this condition automatically holds.

(c) By the reasoning laid out above, an allocation that involves $S_2 < \theta_1 - \theta_2$ can be optimal if and only if $V_0 + M \leq 2r\theta_1 + (\theta_1 - \theta_2)$. Comparing (14) and (15) at $\Gamma_0 = M$ (i.e. $S_1 = S_2 = 0$) leads to that the optimal allocation plan involves zero subsidy and $\Gamma_0 = M$ if and only if $\frac{r}{\theta_2} - \frac{2r-1}{\theta_1+\theta_2} \leq \frac{r}{V_0+M}$. We rewrite the inequality, and obtain $V_0 + M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$. So far we found that such an optimal allocation plan of case (c) requires V_0+M be subject to two upper bounds, $2r\theta_1 + (\theta_1 - \theta_2)$ and $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$. We now compare the two upper bounds, and we claim the former is strictly greater than the latter. To see that, we have

$$\begin{aligned} & 2r\theta_1 + (\theta_1 - \theta_2) - \frac{r\theta_2(\theta_1 + \theta_2)}{r\theta_1 + (1-r)\theta_2} \\ = \; & \frac{2r^2\theta_1(\theta_1 - \theta_2) + r\theta_1^2 + (1-r)\theta_2(\theta_1 - \theta_2)}{r\theta_1 + (1-r)\theta_2} > 0. \end{aligned}$$

(d) If the condition in (c) (as well as those of (a) and (b)) is not satisfied, we must end up with an optimum with $S_1 = 0, S_2 \in (0, \theta_1 - \theta_2)$, and $\Gamma_0 > M - (\theta_1 - \theta_2)$ by the arguments we have laid out in the proof of Proposition 1(c).

Q.E.D. ■

The amounts of equilibrium subsidies in the optimally designed contest that involves positive subsidies and positive top-up prize are presented in the following corollary.

Corollary 2 *(a) When $V_0 < r(\theta_1 + \theta_2 + M)$ and $V_0 + M \geq 2r\theta_1 + (\theta_1 - \theta_2)$, the principal allocates subsidies $S_1^* = \frac{(V_0+M)-(1+2r)\theta_1+\theta_2}{2(1+r)}$ and $S_2^* = \frac{(V_0+M)+\theta_1-(1+2r)\theta_2}{2(1+r)}$, respectively, to Firm 1 and Firm 2.*

(b) When $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} < V_0 + M < 2r\theta_1 + (\theta_1 - \theta_2)$*, the principal allocates a subsidy* $S_2^* = \frac{\sqrt{A_1^2+4A_2}-A_1}{2}$ *to Firm 2 only, where* A_1 *and* A_2 *are as defined before Corollary 1.*

Proof. See Appendix B. ■

The equilibrium in this case exhibits similar properties to that in the case of severe asymmetry. When the amount of private benefit is sufficiently small, i.e. $V_0 \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} - M$, firms would not have sufficient incentive to conduct this research. Hence, all the money should take the form of a top-up prize in order to elicit the desired response. A greater private benefit strengthens firms' incentive to supply their effort, and therefore some money can be spared to subsidize the firms. When the patent value is sufficiently large, i.e., $V_0 \geq r(\theta_1 + \theta_2 + M)$, the principal does not need to provide an additional prize incentive, while it prefers to subsidize the firms to increase their productivity.

Symmetric Firms: A Limiting Case

This model directly applies to the limiting case that involves symmetric firms, with $\theta_1 = \theta_2 = \theta$. In this case, the optimal allocation plan must either involve zero subsidy, or must equally subsidize the two firms. An allocation plan depicted by Proposition 2(d) could never emerge as the optimum. Proposition 2 is therefore adapted to obtain the following.

Corollary 3 *When* $\theta_1 = \theta_2 = \theta$*, in the unique subgame perfect equilibrium, the principal*

(a) allocates the entire budget to subsidize the two firms, i.e., $S_1^* = S_2^* = \frac{M}{2}$ *and* $\Gamma_0^* = 0$ *if and only if* $V_0 \geq r(2\theta + M)$*;*

(b) subsidizes both firms and creates a positive prize, i.e., $S_1^* = S_2^* = \frac{(V_0+M)-2r\theta}{2(1+r)}$ *and* $\Gamma_0^* = \frac{r(2\theta+M)-V_0}{1+r}$ *if and only if* $V_0 < r(2\theta + M)$*, and* $V_0 + M > 2r\theta$*.*

(c) allocates the entire budget to the prize, i.e., $\Gamma_0 = M$*, if and only if* $V_0 + M \leq 2r\theta$*.*

4 Discussion

The main results are first summarized in the following table.

Parameters	Optimal Contest
The Case of Severe Asymmetry $(\theta_1 - \theta_2 > M)$	
$V_0 \geq \frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}$	$S_1^* = 0,\ S_2^* = M,\ \Gamma_0^* = 0$
$V_0 < \frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}, V_0 + M > \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$	$S_1^* = 0,\ S_2^* > 0,\ \Gamma_0^* > 0$
$V_0 + M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$	$S_1^* = 0,\ S_2^* = 0,\ \Gamma_0^* = M$
The Case of Mild Asymmetry $(\theta_1 - \theta_2 \in [0, M])$	
$V_0 \geq r(\theta_1 + \theta_2 + M)$	$S_1^* = \frac{M-(\theta_1-\theta_2)}{2}, S_2^* = \frac{M+(\theta_1-\theta_2)}{2}, \Gamma_0^* = 0$
$V_0 < r(\theta_1 + \theta_2 + M), V_0 + M > 2r\theta_1 + (\theta_1 - \theta_2)$	$S_1^* = S_2^* - (\theta_1 - \theta_2) > 0, \Gamma_0^* > 0$
$\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} < V_0 + M \leq 2r\theta_1 + (\theta_1 - \theta_2)$	$S_1^* = 0, S_2^* > 0, \Gamma_0^* > 0$
$V_0 + M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$	$S_1^* = S_2^* = 0, \Gamma_0^* = M$

Our analysis concerns itself with the optimal allocation of the budget M that minimizes innovation time. When more resources are available, the principal has additional flexibility in designing the incentive structure. This leads to the question of how the principal would reallocate its resources when it has a deeper pocket. In particular, would all the three elements in its bundle $(S_1^*, S_2^*, \Gamma_0^*)$ be assigned more resources?

Proposition 3 $S_1^*(M), S_2^*(M), \Gamma_0^*(M)$ *weakly increase with* M.

Proof. See Appendix. ■

Proposition 3 shows that all the three instruments are considered to be "normal goods" to the principal. When the resources available to the principal increases, they would never decrease the amount of resources allocated to any of the three instruments. Proposition 3 follows Lemmas 1 and 2 and further elaborates upon the roles played by these strategic instruments. In particular, it implies that an equilibrium involving a corner solution (i.e., zero resources on certain instruments) emerges more often if the resources are scarce, while a more balanced mix between subsidies and a prize would result when the amount of resources available increases. This result therefore reveals that subsidies and prizes are not perfectly substitutable in nature; although both are catalysts for success, they function through differing channels.

The rest of this section further examines the nature of the optimal contest structure and the role played by these structural elements (strategic instruments). In particular, we discuss how the value of the private benefit, the technological nature of the innovation project and firms' research capacities, would affect the optimal budget allocation profile.

4.1 The Private Benefit (Patent Value)

The optimal budget allocation profile depends critically on the patent value V_0. The results in both cases have consistently exhibited that the patent value V_0 is inversely related to the incidence of subsidies. The logic that underlies this result is straightforward. As implied by Lemma 1, when the winning firm can expect more rewards from a successful innovation, an additional prize would provide less incentive for further effort supply, while a subsidy that tends to amplify the output of these firms would increase its appeal. By a similar logic, the following result is expected.

Proposition 4 *The amounts of subsidies weakly increases with the patent value* V_0.

Proof. See Appendix B. ■

Proposition 4 states that the patent value V_0 not only increases the frequencies of positive subsidies, but also increases the amount of resources allocated to subsidies in the equilibrium.

An additional prize incentive is required only if a patent itself cannot adequately motivate these firms' innovative activities. The principal therefore resorts to a top-up prize to compensate the lack of this incentive. As directly revealed by our results, an equilibrium where subsidies exhaust the entire budget could emerge if and only if the size of the private benefit is sufficiently large. By way of contrast, when V_0 is sufficiently low such that it falls below the threshold $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} - M$, the contribution from an additional top-up prize completely outweighs that from subsidies, which leads to an equilibrium where no subsidy is given away.

Our results shed light on the design of the incentive mechanism to motivate research efforts. One direct implication is that the optimal incentive structure could crucially depend on the versatility (the scope of application) or the commercializability of the targeted technology. Research firms would have weaker incentive to develop a dedicated military technology without a generous prize purse awaiting the winner. This could explain the observation that DoD intensively seeks a prize incentive, such as procurement contracts, to motivate innovations that are often dedicated to military applications. By way of contrast, governments and non-profit organizations such as the United Nations frequently dispense scarce resources in seeking medical cures or vaccines to limit the spread of deadly diseases. In contrast to dedicated military technology, the broader application of civil medical research leads to substantial profitability. Hence, in order to speed up the delivery of a medical discovery, priority could be more frequently given to research subsidies towards pharmaceutical research entities instead of luring them through procurement contracts.

Our analysis also yields useful insights to the long lasting debate on patent policy. Strong patent protection fairly rewards intellectual property right owners, which tends to promote innovative effort; while it could inhibit free flow of scientific knowledge and prevent innovations from being maximally utilized. Its adverse effect could particularly loom large for an innovation that generates substantial social benefits other than its reward to its owner, e.g., an effective medical cure for communicable disease. Our analysis may provide one alternative to balance private incentives and social benefit under such a circumstance: to internalize the positive externality, the government may consider weakening the patent protection on such innovation, while compensate for the lack of private incentive by providing generous procurement contract to successful innovators.

4.2 The Technological Nature of the Innovation: The Role of "r"

The technological nature of the innovation is depicted by the parameter r. The parameter literally measures the elasticity of the hazard rate to additional effort, i.e., how much incremental effort could contribute to the likelihood of success. Hence, a greater r implies that the success of the innovation relies more heavily on continuing effort, rather than on a sudden spurt of inspiration. A lower r thus reflects that more uncertainty or difficulty is involved in the development of the targeted technology.

A casual look at our results reveals that the properties of the optimal contest strongly depend on the magnitude of r. In general, the principal is more likely to provide a "top-up prize" (procurement contract) when the success of the project is more sensitive to additional effort, i.e., a greater r. This argument is demonstrated by analyzing the critical values of V_0 that define differing equilibria.

The case of severe asymmetry is first considered, i.e., $\theta_1 - \theta_2 > M$. An equilibrium where the subsidy S_2 exhausts the budget would emerge when V_0 exceeds the boundary $\frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}$. Rewrite the boundary as $\frac{(\theta_1+\theta_2+M)(\theta_2+M)}{\theta_1+\frac{(1-r)}{r}(\theta_2+M)}$, and it can be seen to strictly increase with r. Thus, the condition that leads to zero top-up prize (Proposition 1(a)) is less likely to be met when r increases. In contrast, the top-up prize exhausts the budget (Proposition 1(b)) when $V_0 \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} - M$. Again, a greater r lifts this bar, as $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$ strictly increases with r. The parameter r plays a similar role in the case of mild asymmetry, i.e., $\theta_1 - \theta_2 \leq M$. Thus, it can be concluded that when r increases, i.e., when the innovation is more sensitive to additional effort input, the incidence of positive subsidies would fall in response, while a prize incentive emerges more often.

Proposition 5 *The equilibrium amounts of subsidies weakly decrease with r, i.e., the elasticity of the hazard rate to additional effort.*

Proof. See Appendix. ■

Proposition 5 states that the amounts of subsidies increase with the level of uncertainty involved in the innovation project. The result of Proposition 5 is illustrated in Figure 2,

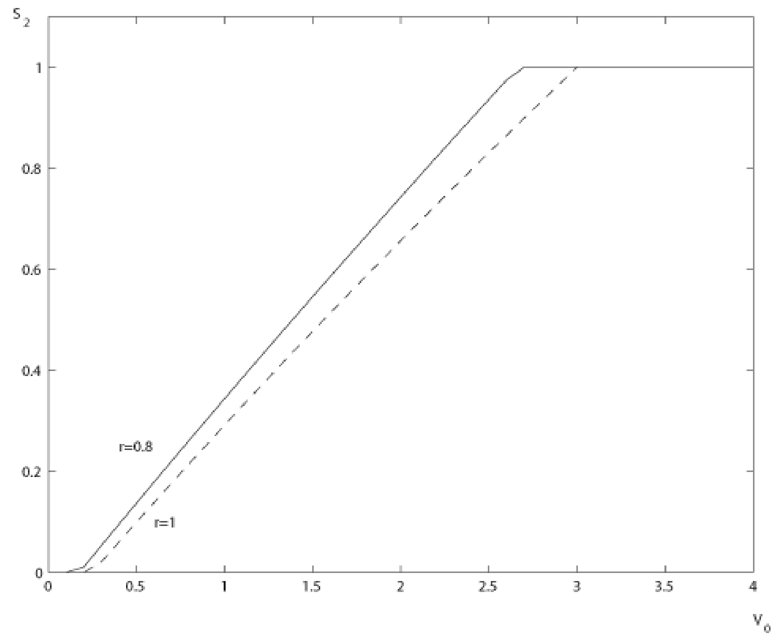

Figure 2: The Impact of r on S_2^*

which depicts in case one (severe asymmetry) the equilibrium subsidy to Firm 2. The values for the parameters are as follows: $\theta_1 = 4$, $\theta_2 = 1$, $V_0 = 1$, $M = 1$. The curve that plots S_2^* is consistently shifted down when r rises from 0.8 to 1.0. Similar patterns can be observed when both S_1^* and S_2^* are plotted for other cases.

The implications of this result are directly revealed by the expression of equilibrium effort outlay as given by (7). The equilibrium effort outlay is linear in both the prize purse and the elasticity parameter r. A greater r amplifies the incentive provided by the winner prize, as it directly enlarges the marginal impact of effort on winning chance. Thus, a greater r heightens the appeal of offering a prize incentive.

The effectiveness of an instrument critically depends on the level of uncertainty involved in the development process. When there is tremendous uncertainty about the actual effectiveness of the R&D activity, competing parties would hold out lesser prospects of eventual success. A prize, however, can be won only after the uncertainty involved in the innovative activities is resolved. Hence, a "prize" that is contingent on success would be heavily discounted by rational firms, and cannot effectively motivate their effort. As a result, research subsidies that enhance the parties' capabilities would offer more compelling incentives as subsidies are given away before the uncertainty is resolved, which tends to mitigate the uncertainty.

This result sheds light on the design of incentive mechanisms to promote scientific discovery. The choice of instruments depend on the nature of the innovation being pursued. A generous prize purse could successfully lure competitive effort for an industrial technical solution that is not extremely challenging. However, it cannot elicit substantial effort towards breakthrough in fundamental scientific theory. As widely observed in reality, a science foundation that aims to foster advancement in basic research usually provides incentives in the form of financial support to scientists more often than contingent monetary rewards. The enormous uncertainty associated with fundamental research project diminishes the additional incentive a prize could provide. It is thus more efficient to provide a research grant that improves scientists' research capacity.

4.3 Firms' Ex Ante Research Capacities

The results of this paper have confirmed the limited substitutability of awarding a prize to the winning firm compared to providing subsidies or aid to firms. As shown by Lemma 1 and the thought experiment in Section 3, a parallel increase in a firm's efficiency parameters θ_is (which does not affect the balance in the competition) reduces the need to improve its capability. A more generous procurement contract would thus have greater appeal. This limited substitutability therefore enables the principal to redirect its resources toward the prize purse.

It has been shown that a subsidy to a firm increases (decreases) effort outlays at equilibrium if and only if it balances (unbalances) the playing field. Thus, the principal has to preferentially subsidize the weaker firm when it balances the subsidies between firms.[19] This result echoes the logic espoused in the literature on efficient handicapping, such as that demonstrated in the works of Che and Gale (1998 and 2003), Clark and Riis (2000) and Fu (2006).

As previously mentioned, our results provide a rationale for the support given, in the form of lending, by the U.S. government and others to small businesses that conduct innovative research. A more level playing field encourages weaker firms to step up their efforts, which in turn discourages stronger firms from slackening. Our analysis therefore reveals the positive effect of a preferential funding system in favor of small businesses: not only does such a system allow small firms to engage more actively in innovation, it also disciplines bigger firms and incentivizes them to increase their efficiency.

A few remarks are in order. **First**, we obtained our results from an analysis of a stylized two-firm setting. Nonetheless, the logic is still valid in more extensive settings, e.g. when

[19] According to equation (27) in the Appendix, a redistribution of the total amount of the subsidy between the two firms does not have a direct impact on the expected time taken to develop an innovation. Instead, the redistribution of resources affects the expected time required to develop the innovation through the equilibrium effort x^*.

more than two firms are involved in the competition.

To check the robustness of our result, we consider the (simplified) case of a three-firm race in Appendix A. The result is largely consistent with our observations of the two-firm setting. This extension complements our current analysis and improves our understanding of the main results.

When three firms with different capabilities compete in a race, and one firm is excessively disadvantaged against the other two, the weakest firm may stay inactive (i.e., exert zero effort) in the equilibrium. The possibility of a firm remaining inactive greatly complicates the principal's budget allocation problem. A subsidy allocation rule could serve as an incentive-compatible mechanism that allows the principal to select participating (active) firms so as to optimize the structure of the subsequent competition. The principal may either exclude an otherwise active firm from the race by heavily subsidizing the others, or revive an otherwise inactive firm by preferentially subsidizing it.

Again, we find that in the three-firm case, the optimum always involves the preferential subsidization of the weakest of the **active** firms. The strongest firm never receives the highest subsidy. This observation builds on the same logic we have demonstrated in the two-firm setting: A more level playing field creates more competition. However, the amount of subsidy each firm receives may not be monotonic in their initial ranks. The weakest firm does not necessarily receive more subsidy than the others, because the principal may prefer to exclude the weakest by concentrating its resources on the others. Although subsidizing the weakest firm could balance the playing field and create more competition in a given three-firm race, it appropriates the limited resource that can be utilized otherwise. Instead, the principal may prefer a two-firm race, and the firm in the middle receives the highest subsidy to achieve this end. Concentrating resources on the two favorites leads to more competent participating firms and increases the overall quality of the race. Preferentially subsidizing the firm in the middle also allows for a more even race between the two favorites. A number of forces could lead to such an optimum, but a more complete account of this problem is analytically difficult and is outside of the central interest of the current paper. Nonetheless, Appendix A presents several illustrative examples of this possibility which allows us to gain further insights into our main results.

Second, it should be noted that the competition depicted by our model is not entirely similar to the competition involving scientific grant applications (e.g., NSF grant applications). Grant applicants compete by the quality of their proposals, while grants are given away as "prizes" that reward the strongest proposals. However, in the R&D contest we have considered, firms race towards a specific innovation. The winner, who successfully implements the given innovation, is rewarded by the award of a lucrative patent, as well as by the top-up prize provided by the principal. Instead of being merit based, research subsidies are provided non-contingently as a means of facilitating the firms' eventual success.

Nevertheless, the optimal subsidy allocation rule we find in the current scenario does not conflict with the "meritocratic" selection mechanism often perceived in grant competitions. As we observe from the three-firm scenario, the principal may exclude a weak participant, and choose instead to provide subsidies only to the more "qualified" (top two) candidates. Further, the "handicapping" rule characterized in our model appears in the grant application process as well. Many science foundations have set aside funds dedicated to the applications of young researchers. For instance, the first funding stream of European Research Council, which accounts for one third of its budget, was announced to be available to young researchers only.

5 Concluding Remarks

This paper has studied the optimal design of research contests. The principal was allowed to design the contest using two strategic vehicles: subsidies to competing firms and a top-up prize (procurement contract). The principal faced a budget constraint and her objective was to minimize the expected amount of time required for an innovation. It was found that, when firms differ in their initial capabilities, the principal can effectively speed up the innovation by preferentially subsidizing the weaker firm. Furthermore, an additional prize incentive (procurement contract) occurs more often when less uncertainty is involved in the innovative project, while it occurs less frequently when the successful innovator receives a greater private benefit from the patent.

Analytical tractability and expositional efficiency have limited our analysis in a stylized setting. We now discuss possible extensions, including the case of financial constrained firms, and the case where firms discount future payoffs. We show that while additional insights obtain in the more extensive setting further unveil the logic underlying our findings, the main results of our analysis are in general robust and largely extend to these contexts.

5.1 Financially Constrained Firms

Financial constraints are not explicitly included in this model. Research activities are often impeded by limited resources. Most academics consider research funds to be inadequate to support scientific projects. As noted by Che and Gale (2003), DoD worries about the financial stability of research companies, so it extensively subsidizes invited firms to maintain the competition. It appears natural to assume that firms are constrained by the resources available to them. However, it is worth noting that this model does not lose its bite in this regard. Recall that a firm's ex ante technology in our model is given by $h_i(x_i) = \theta_i x_i^r$.

A binding financial constraint could function with the equivalence of a reduced capacity θ_i. Both a lower θ_i and a binding constraint tend to restrict the firms' actual equilibrium effort input. Hence, there is no loss of generality to presume that the generic measure θ_i of capacity reflects the firms' financial adequacy. Thus, when binding resource constraints are present, the results discussed in Section 3.2 allow us to predict increasing incidences of subsidies against a prize incentive.

5.2 When Firms Discount Future

In the current setting, it is explicitly assumed that firms do not discount future payoff. As pointed out in Section 2, this setting, although limited, could mirror a situation where the principal's interest is not well aligned with that of the firms, such that the desired innovation creates more imminent benefit to the principal than it does to the firms. If the firms also discount future payoff at a discount rate of $\widehat{\rho} \in (0, \rho)$, a firm i's expected payoff is thus described as

$$\begin{aligned} \pi_i(x_i, x_j) &= \int_0^{\infty} e^{-\widehat{\rho} t} f_i(t|x_i)(1 - F_j(t|x_j)) \cdot (V_0 + \Gamma_0)\, dt - x_i \\ &= \frac{h_i(x_i)}{h_i(x_i) + h_j(x_j) + \widehat{\rho}} (\Gamma_0 + V_0) - x_i. \end{aligned} \tag{22}$$

Simple rationale reveals that a nonzero discount factor could affect the structure of the optimal contest. The discount rate $\widehat{\rho}$ plays a qualitatively opposite role to that of the elasticity parameter r. A greater $\widehat{\rho}$ diminishes the impact of any given prize incentive. As a result, the resource that is allocated to the top-up prize would be less effective in motivating these firms. Thus, returning to the results laid out in Section 3.2, when firms are more eager, the incidence and amounts of subsidies can be expected to increase. Alternatively, more eager firms imply that the interests of the principal and the firms are better aligned, which weakens the need for a prize incentive to elicit extra effort. Subsidies that amplify firms' given outputs thus have a greater appeal.

When $\widehat{\rho}$ remains moderate, the presence of this factor would not qualitatively vary the main results. However, additional complexity could result from a huge $\widehat{\rho}$. When $\widehat{\rho}$ is excessively large, it may no longer be optimal to preferentially subsidize the ex ante weaker firm. Looking at the payoff function (22), the discount factor can be interpreted as a third competitor in this race. When only two firms are engaged, preferentially subsidizing the weaker firm prevents the stronger one from slackening. However, this effect may disappear when $\widehat{\rho}$ is excessively large. As will be pointed out by the discussion in Appendix A, it could be more efficient, then, in that case to preferentially subsidize the stronger firm. The stronger firm exerts more productive effort, and an additional subsidy could further assist it to "race against the clock".

In summary, it should be noted that the above results apply in settings with moderately sized $\widehat{\rho}$. In order to view the panorama in the presence of a large $\widehat{\rho}$, additional analysis would be required, despite its analytical difficulty.

Appendix A: A Case of Three-Firm Competition

To better grasp the logic that underlies our results, we now consider a scenario in which three firms participate in an R&D race. For the sake of tractability, we assume that each firm possesses linear technology, i.e., $r = 1$.[20]

We first derive the equilibrium in the race. Denote by $\tilde{\theta}_i$ a firm i's ex post capacity measure with $\tilde{\theta}_i = \theta_i + S_i$ where $S_i (\geq 0)$ is the subsidy to i. The payoff of i is $\pi_i = \frac{\tilde{\theta}_i x_i}{\sum_j \tilde{\theta}_j x_j} V - x_i$ where $V = V_0 + \Gamma_0$ and x_j is the effort of j. The first order derivatives $\frac{\partial \pi_i}{\partial x_i}$ are:

$$\frac{\partial \pi_i}{\partial x_i} = \frac{\tilde{\theta}_i \sum\limits_{j \neq i} \tilde{\theta}_j x_j}{(\sum\limits_{j=1}^{3} \tilde{\theta}_j x_j)^2} V - 1, \forall i \in \{1, 2, 3\}. \tag{23}$$

Note that $\frac{\partial \pi_i}{\partial x_i}$ decreases with x_i.

Suppose $\tilde{\theta}_k (> 0)$ is the smallest among all $\tilde{\theta}_i$s. We have the following Lemma.

Lemma 4 *(a) If* $\tilde{\theta}_k > \frac{\prod_{j \neq k} \tilde{\theta}_j}{\sum_{j \neq k} \tilde{\theta}_j}$*, then* $\tilde{\theta}_i > \frac{\prod_{j \neq i} \tilde{\theta}_j}{\sum_{j \neq i} \tilde{\theta}_j}, \forall i$*. (b) If* $\tilde{\theta}_k \leq \frac{\prod_{j \neq k} \tilde{\theta}_j}{\sum_{j \neq k} \tilde{\theta}_j}$*, then* $\tilde{\theta}_i > \frac{\prod_{j \neq i} \tilde{\theta}_j}{\sum_{j \neq i} \tilde{\theta}_j}, \forall i \neq k$*.*

Proof. See Appendix B. ■

We conclude the following.

Proposition 6 *(a) A unique equilibrium with* $x_i^* > 0$, $\forall i$ *exists if* $\tilde{\theta}_k > \frac{\prod_{j \neq k} \tilde{\theta}_j}{\sum_{j \neq k} \tilde{\theta}_j}$ *where firm* i *invests an equilibrium effort*

$$x_i = \frac{a}{\tilde{\theta}_i} - \frac{1}{V} \left(\frac{a}{\tilde{\theta}_i} \right)^2, \tag{24}$$

where $a = 2V / (\frac{1}{\tilde{\theta}_1} + \frac{1}{\tilde{\theta}_2} + \frac{1}{\tilde{\theta}_3})$.

(b) A unique equilibrium with $x_k^* = 0$ *exists if* $\tilde{\theta}_k \leq \frac{\prod_{j \neq k} \tilde{\theta}_j}{\sum_{j \neq k} \tilde{\theta}_j}$ *where the other two firms invest an equilibrium effort* $x^* = \frac{\prod_{j \neq k} \tilde{\theta}_j}{(\sum_{j \neq k} \tilde{\theta}_j)^2} V$.

[20] When $r \neq 1$, the equilibrium, if it exists, would be determined by a set of higher-order equations. Closed-form solutions generally do not exist.

Proof. See Appendix B. ■

Proposition 6 provides a closed-form solution to the equilibrium in the three-firm case. It shows that all firms stay active (i.e., they exert positive effort) if and only if the weakest is not excessively disadvantaged. If this condition is not met, the basic setting described in Section 2 will be restored such that only the two favorites compete against each other.

The solution provided by Proposition 6 provides further insight into the problem of optimal budget allocation. The nature of the trade-off between the prize and the subsidies will not be fundamentally altered by the addition of a third participating firm. However, this addition complicates the principal's problem of allocating subsidies among the firms.

Without a loss of generality, assume that the three firms are initially ranked as $\theta_1 \geq \theta_2 \geq \theta_3$. Recall that the principal sets out to minimize $E(t)$, which is equivalent to maximizing the aggregate output $\tilde{Y} \equiv \sum_{i=1}^{3} \tilde{\theta}_i x_i$. We first look at the results of comparative statics to highlight the local properties of the normalized total output function $\frac{\tilde{Y}}{V}$.[21] Consider a given subsidy allocation rule (S_i). When $\tilde{\theta}_3 \geq \prod_{j \neq 3} \tilde{\theta}_j \diagup \sum_{j \neq 3} \tilde{\theta}_j$, the first order conditions delivers $\tilde{Y} = a = \frac{2V}{\frac{1}{\tilde{\theta}_1} + \frac{1}{\tilde{\theta}_2} + \frac{1}{\tilde{\theta}_3}}$.[22] We thus have

$$\frac{\partial \tilde{Y}}{\partial \tilde{\theta}_i} / V = \frac{2 \prod_{j \neq i} \tilde{\theta}_j^2}{\left(\prod_{h \neq j} \tilde{\theta}_h \tilde{\theta}_j \right)^2} > 0, \forall i. \tag{25}$$

We can observe from (25) that $\frac{\partial \tilde{Y}}{\partial \tilde{\theta}_i} / V \gtreqless \frac{\partial \tilde{Y}}{\partial \tilde{\theta}_j} / V$ if and only if $\widetilde{\theta}_i \lesseqgtr \widetilde{\theta}_j$. This implies that an infinitesimal amount of additional subsidy would better catalyze the innovation when a less efficient firm receives it. This observation confirms that it is efficient to provide a greater subsidy to the weaker firm. Although the logic of Lemma 2 continues to apply, its limits must be more thoroughly examined.

When three firms are included in the analysis, another important strategic concern is added to the principal's subsidy allocation problem. A subsidy not only increases the recipient firm's marginal output and mediates the balance between firms, but may also vary the structure of the competition globally. Note that the cutoff for the weakest firm staying active, $\prod_{j \neq k} \tilde{\theta}_j \diagup \sum_{j \neq k} \tilde{\theta}_j$, strictly increases with $S_j, \forall j \neq k$. The principal may heavily subsidize

[21] For purposes of the current analysis, it suffices to examine the properties of the normalized total output. We do not consider how the principal decides to make the trade-off between the amount allocated to the provision of subsidies, and to the prize. Instead, we investigate only how subsidies should be allocated among participating firms.

[22] The first order conditions can be written as $\tilde{Y} - \tilde{\theta}_i x_i = \frac{\tilde{Y}^2}{V \tilde{\theta}_i}, \forall i$. Summing up across i, we have $2\,\tilde{Y} = (\frac{\tilde{Y}^2}{\tilde{\theta}_1} + \frac{\tilde{Y}^2}{\tilde{\theta}_2} + \frac{\tilde{Y}^2}{\tilde{\theta}_3})\frac{1}{V}$.

a subset of these firms to drive one firm out of the race. In contrast, subsidizing an initially inactive firm may bring it back to the competition. By manipulating subsidy allocation among firms, the principal can either expand the size of the competition, or narrow the set of active contenders.

We now establish the global properties of the optimal resource allocation plan.

Proposition 7 *When $\theta_1 \geq \theta_2 > \theta_3$, an optimal allocation plan requires that (1) firm 1 and firm 2 stay active; (2) $S_2^* \geq S_1^*$ while either $S_3^* \geq S_2^*$ or $S_3^* = 0$; and (3) $\widetilde{\theta}_1^* \geq \widetilde{\theta}_2^* \geq \widetilde{\theta}_3^*$.*

Proof. See Appendix B. ■

Proposition 7 states that the optimum must include the two ex ante most efficient firms in the race. Analogous to the two-firm setting, the principal never subsidizes the most efficient firm to a greater level than it subsidizes the second most efficient firm. However, the preferential allocation plan cannot reverse the initial asymmetry among the firms. Hence, we show that our main result continues to hold in this setting: **Regardless of the number of active firms, the optimal contest must provide the highest subsidy to the least efficient active firm**!

Although the optimum continues to favor the second most efficient firm over the first, the least efficient firm does not necessarily receive more subsidy than its ex ante more efficient rivals. The firm in the middle may instead receive the highest aid.[23] We illustrate this argument using the following example.

Example 1 *Let firm 3's initial capacity measure θ_3 falls below the cutoff $\frac{\theta_1 \ \theta_2}{\theta_1 \ +\theta_2}$. When the budget M is limited, i.e., $M \leq \frac{\theta_1 \ \theta_2}{\theta_1 \ +\theta_2} - \theta_3$, we always have $\widetilde{\theta}_3 \leq \frac{\widetilde{\theta}_1 \ \widetilde{\theta}_2}{\widetilde{\theta}_1 \ +\widetilde{\theta}_2}$ and firm 3 stays inactive regardless. From this scenario, it is clear that no subsidy should be provided to firm 3.*

Firm 3 is "naturally" excluded from the competition, as the principal cannot bring it back to the race. This implies that subsidizing firm 3 may not pay off, even if the firm is not "naturally" excluded. According to the continuity argument, when $\theta_3 < \frac{\theta_1 \ \theta_2}{\theta_1 \ +\theta_2}$ and M is slightly above the cutoff $\frac{\theta_1 \ \theta_2}{\theta_1 \ +\theta_2} - \theta_3$, the principal has to devote virtually all M to S_3. It does not elicit significantly more effort because $\frac{\partial \tilde{Y}}{\partial \widetilde{\theta}_3} = 0$ whenever $\widetilde{\theta}_3 < \frac{\widetilde{\theta}_1 \ \widetilde{\theta}_2}{\widetilde{\theta}_1 \ +\widetilde{\theta}_2}$. For this reason, it could be too costly to bring a firm of very low caliber to a level that sufficiently threatens and disciplines its more efficient rivals.

The next example demonstrates that it can be optimal to exclude an otherwise active firm 3 by preferentially subsidizing the others.

[23] In this case, the subsidy to firm 3 must be zero such that this firm is rendered inactive. Please see the proof of Proposition 7 for details.

Example 2 *Consider a three-firm race with* $\theta_3 = \frac{\theta_1\ \theta_2}{\theta_1\ +\theta_2}$*. Let* θ_1 *be substantially advantaged against the others, with* $\theta_1 > (1+\sqrt{2})\theta_2$*. Suppose also that* V_0 *is sufficiently high, such that a small resource value is optimally allocated as a subsidy to one of the competing firms. If the subsidy amount is allocated to firm 3, all firms remain active, and the normalized total output* $\frac{\tilde{Y}}{V}$ *increases at a rate* $\frac{\partial\ \tilde{Y}}{\partial\ \tilde{\theta}_3}/V\Big|_{\tilde{\theta}_3=\frac{\theta_1\ \theta_2}{\theta_1\ +\theta_2}} = \frac{1}{2}$*. If it is allocated to firm 2, a two-firm race emerges, and the normalized total output* $\frac{\tilde{Y}}{V}$ *increases at a rate* $\frac{\partial\ \tilde{Y}}{\partial\ \tilde{\theta}_2}/V\Big|_{\tilde{\theta}_2=\theta_2} = \frac{\theta_1^2}{(\theta_1+\theta_2)^2}$*. Because* $\frac{\theta_1^2}{(\theta_1+\theta_2)^2} > \frac{1}{2}$ *if and only if* $\theta_1 > (1+\sqrt{2})\theta_2$*, it is more efficient to subsidize firm 2. According to the continuity argument, subsidizing firm 2 can continue to be the more efficient option when* θ_3 *is within a relatively small range of* $\frac{\theta_1\ \theta_2}{\theta_1\ +\theta_2}$*.*

Intuitively, excluding firm 3 by preferentially subsidizing the others may be the more appealing option when the distribution of the firms' initial capacities is highly dispersed. This example demonstrates the logic underlying the effect of preferential subsidization. In our example, firm 3 substantially lags behind the others, while firm 1 is excessively strong. Given the significant heterogeneity among the firms and the limited value of the resources available for subsidy allocation, aiding the weakest firm cannot sufficiently tilt the balance of the playing field. However, doing so may serve to crowd out the limited resource and it may be more efficient to preferentially subsidize firm 2, which is the more efficient firm. The principal may then create a more even race between the two favorites to maximize the efficiencies gained from the resources expended. Note that this practice can be optimal even when firms are homogenous.

Example 3 *Consider three symmetric firms with* $\theta_1 = \theta_2 = \theta_3 = \theta$*. Imagine that the budget* M *is big enough such that at the optimum an amount of resource* $\Delta > 6\theta$ *is available to subsidizing the firms. If all firms are to be included in the race, first order conditions imply* $\widetilde{\theta}_1 = \widetilde{\theta}_2 = \widetilde{\theta}_3 = \theta + \frac{\Delta}{3}$*, which leads to total output* $\tilde{Y} = \frac{2\ (\theta+\frac{\Delta}{3})}{\mathbf{3}}V$*. However, the principal may evenly split* Δ *between firm 1 and firm 2. This allocation plan would drive firm 3 out of the race, because* $\frac{\widetilde{\theta}_1\widetilde{\theta}_2}{\widetilde{\theta}_1+\widetilde{\theta}_2} = \frac{(\theta+\frac{\Delta}{2})^2}{2\theta+\Delta} = \frac{\theta+\frac{\Delta}{2}}{2} > 2\theta$*. The two-firm race yields a total output* $\tilde{Y} = \frac{\theta+\frac{\Delta}{2}}{2}V$*, which is greater than that of three active firms if and only if* $\Delta > 6\theta$*. Hence, excluding one firm by subsidizing the others is optimal.*

Example 3 shows that when (symmetric) firms are sufficiently inefficient, i.e., $\theta < \frac{\Delta}{6}$, it does not benefit the principal to retain all of them in the race. Instead, the principle prefers to maximally improve the caliber of only two of the three firms such that they perform more optimally. A two-firm race is thus restored.

The examples provided testify to the potential value of excluding the weakest firm from the optimum scenario. They suggest that the optimal exclusion strategy depends on the interaction among the amount of budget available to the principal, the distribution of the firms' initial endowments, and the total amount of private benefit the winner can potentially receive.[24] A complete characterization of the optimal exclusion strategy is analytically difficult, and beyond the scope of this paper. However, the examples illustrate the insights of Proposition 7 and allow us to interpret our main results with greater certainty.

These observations provide interesting contrast to several studies in the shortlisting literature. Baye, Kovenock and de Vries (1993) showed that it can be optimal to exclude the most efficient bidder in an all-pay auction when the contest organizer is allowed to shortlist bidders. Taylor (1995) showed the benefit of exclusion to the contest sponsor when entry to a research tournament implies a fixed cost. Che and Gale (2003) similarly established the optimality of shortlisting two firms to participate in a R&D tournament. However, none of these studies involved the positive subsidization of selected firms to exclude otherwise active firms. It deserves to be noted that shortlisting (without manipulating subsidy allocation) can never be optimal in our setting when $\theta_3 > \frac{\theta_1\theta_2}{\theta_1+\theta_2}$. The three-firm race yields a total output $\tilde{Y} = \frac{2\theta_1\theta_2\theta_3}{\theta_1 \ \theta_2 + \ \theta_1\theta_3+\theta_2\theta_3}V$. If the principal shortlists two participating firms, firm 1 and firm 2 must be shortlisted.[25] The total output would be given by $\tilde{Y}' = \frac{\theta_1\theta_2}{\theta_1+\theta_2}V$. Hence, we have $\tilde{Y}/\tilde{Y}' = \frac{\theta_1\theta_3+\theta_2\theta_3+\theta_1\theta_3+\theta_2\theta_3}{\theta_1 \ \theta_2+ \ \theta_1\theta_3+\theta_2\theta_3} > 1$, because $\theta_1\theta_3 + \theta_2\theta_3 > (\theta_1 + \theta_2)\frac{\theta_1\theta_2}{\theta_1+\theta_2} > \theta_1\theta_2$. However, as illustrated in Example 3, when the principal is allowed to subsidize selected firms, exclusion can turn out to be optimal. Nevertheless, as shown by Proposition 7, an optimum that excludes ex ante more (strictly) efficient firms does not exist. Our results therefore run in contrast to Baye, Kovenock and de Vries (1993).

At the same time, contrary also to the findings of Che and Gale (2003), excluding the weakest firm 3 is not necessarily optimal. As implied by Example 3, when these firms are ex ante nearly homogeneous, and the budget is very limited, the principal will not exclude any firm from the race.

6 Appendix B: Proofs

Proof of Lemma 1

Proof. Since $r \leq 1$, we have $\frac{2r-1}{\theta_1+\theta_2+S_1+S_2} - \frac{r}{\theta_i+S_i} \leq \frac{r}{\theta_1+\theta_2+S_1+S_2} - \frac{r}{\theta_i+S_i} < 0$. Clearly, $\frac{\partial E(t)}{\partial \Gamma_0} < 0$. Now we show $\frac{\partial^2 E(t)}{\partial S_i^2} > 0$ and $\frac{\partial^2 E(t)}{\partial \Gamma_0^2} > 0$, i.e., their marginal impact decreases. To show $\frac{\partial^2 E(t)}{\partial S_1^2} > 0$, we need to show $\frac{\partial \left|\frac{\partial E(t)}{\partial S_1}\right|}{\partial S_1} < 0$. Because $\left|\frac{\partial E(t)}{\partial S_1}\right| = E(t)\,\xi_1$, it suffices to show ξ_1 decreases with S_1.

[24] How much should be allocated in subsidies determines together with other factors the amount of private benefit the winner can receive.

[25] Shortlisting firm 1 and firm 2 leads to the highest equilibrium total output. Please refer to the proof of Proposition 7 for details.

Simple calculus reveals that $\frac{\partial \xi_1}{\partial S_1} = \frac{2r-1}{(\theta_1+\theta_2+S_1+S_2)^2} - \frac{r}{(\theta_1+S_1)^2} \leq \frac{r}{(\theta_1+\theta_2+S_1+S_2)^2} - \frac{r}{(\theta_1+S_1)^2} < 0$. The marginal impacts of S_2 and Γ_0 can be similarly shown.

Q.E.D. ■

Proof of Lemma 2

Proof. We imagine that an infinitesimal amount of additional money is to be given to firm i. Since $E\left(t|x^*\right) = \frac{1}{((\theta_1+S_1)+(\theta_2+S_2))x^{*r}}$, its marginal impact on $E\left(t|x^*\right)$ is given by

$$\frac{\partial E\left(t\right)}{\partial S_i} = \frac{\partial E\left(t|x^*\right)}{\partial S_i} + \frac{\partial E\left(t|x^*\right)}{\partial x^*} \cdot \frac{\partial x^*}{\partial S_i}, \tag{26}$$

which is a composition of a direct effect

$$\frac{\partial E\left(t|x^*\right)}{\partial S_i} = \frac{-x^{*r}}{\{[(\theta_i + S_i) + (\theta_j + S_j)]x^{*r}\}^2}, \tag{27}$$

and an indirect effect $\frac{\partial E(t|x^*)}{\partial x^*} \cdot \frac{\partial x^*}{\partial S_i}$.

One observes from (27) that the direct effect of $S_i, i = 1, 2$ always has a negative sign, i.e., the direct effect tends to reduce the expected innovation time, regardless of the identity of the recipient. In addition, the magnitude of this first order impact does not depend on the identity of the recipient of subsidy. However, the sign of the indirect effect is indefinite. Apparently, $E(t)$ strictly decreases with equilibrium effort x^*. By (12), when the stronger (weaker) efficient firm is further subsidized, the equilibrium effort decreases (increases). This implies that when an additional amount of money is available to subsidize firms, it is a strictly dominant strategy to allocate it to the currently weaker firm. We therefore conclude that $\theta_1 + S_1^* \geq \theta_2 + S_2^*$ must hold in the equilibrium. That is, the optimal allocation plan cannot reverse the initial asymmetry, which gives rise to Lemma 2(a). Results (b) and (c) immediately follow.

Q.E.D. ■

Proof of Corollary 1

Proof. Two roots given by $\frac{-A_1 \pm \sqrt{A_1+4A_2}}{2}$, can be obtained by solving (18) using standard technique. By Proposition 1(c), a unique $S_2^* \in (0, M)$ exists. $A_2 = (r\theta_1 + (1-r)\theta_2)(V_0 + M) - r\theta_2(\theta_1 + \theta_2) > 0$ by the condition $V_0 + M > \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$. Thus, we must have $\sqrt{A_1^2 + 4A_2} > A_1\,(> 0)$, and the unique positive root is $\frac{\sqrt{A_1^2+4A_2}-A_1}{2}$.

Q.E.D. ■

Proof of Corollary 2

Proof. (a) When an interior equilibrium prevails which involves both firms receiving positive subsidies, proof of Proposition 2(b) shows that the equilibrium requires

$$\frac{1}{2\left(\theta_1+S_1^*\right)}=\frac{r}{V_0+M-\left(\theta_1-\theta_2\right)-2S_1^*}. \tag{28}$$

We thus could obtain the optimal allocation bundle by solving this equation.

(b) In this case, the equilibrium condition is the same as that of Proposition 1(c). Equation (18) continues to apply and we obtain the desirable result.

Q.E.D. ■

Proof of Proposition 3

Proof. For descriptive convenience, we define $\xi_1=\frac{r}{\theta_1+S_1}-\frac{2r-1}{\theta_1+\theta_2+S_1+S_2}>0$, $\xi_2=\frac{r}{\theta_2+S_2}-\frac{2r-1}{\theta_1+\theta_2+S_1+S_2}>0$ and $\xi_3=\frac{r}{V_0+\Gamma_0}>0$. Note that the optimal allocation of the budget solely depends on comparisons among ξ_1,ξ_2 and ξ_3.

Suppose that we have $M_1>M_0$. We claim $S_1^*\left(M_0\right)\leq S_1^*\left(M_1\right), S_2^*\left(M_0\right)\leq S_2^*\left(M_1\right)$, and $\Gamma_0^*\left(M_0\right)\leq\Gamma_0^*\left(M_1\right)$. Define $\Delta\equiv M_1-M_0>0$. Initially the principal has an optimum $\left(S_1^*\left(M_0\right),S_2^*\left(M_0\right),\Gamma_0^*\left(M_0\right)\right)$. Note that we must have $S_2^*\left(M_0\right)\geq S_1^*\left(M_0\right)$ from Lemma 2. We consider the following possible cases.

Case 1: $\Gamma_0^*\left(M_0\right)=0$, $S_1^*\left(M_0\right)=0, S_2^*\left(M_0\right)=M_0$.

In this case we must have $\xi_2^*(M_0)\geq\max\{\xi_1^*(M_0),\xi_3^*(M_0)\}$. Recall that $\xi_1=\frac{r}{\theta_1+S_1}-\frac{2r-1}{\theta_1+\theta_2+S_1+S_2}$, $\xi_2=\frac{r}{\theta_2+S_2}-\frac{2r-1}{\theta_1+\theta_2+S_1+S_2}$, and $\xi_3=\frac{r}{V_0+\Gamma_0}$. In order to verify the claim, we only need to show that $S_2^*\left(M_0\right)\leq S_2^*\left(M_1\right)$.

Suppose $S_2^*\left(M_0\right)>S_2^*\left(M_1\right)$, then we must have $S_1^*\left(M_0\right)<S_1^*\left(M_1\right)$ or $\Gamma_0^*\left(M_0\right)<\Gamma_0^*\left(M_1\right)$. By Lemma 2, $S_1^*\left(M_1\right)$ must remain at zero if $S_2^*\left(M_0\right)>S_2^*\left(M_1\right)$. Thus, we can only have $\Gamma_0^*\left(M_1\right)>\Gamma_0^*\left(M_0\right)=0$. However, this implies $\xi_2^*(M_1)>\xi_3^*(M_1)$, which conflicts with the first order conditions for the optimum.

Case 2: $\Gamma_0^*\left(M_0\right)=0$, $S_1^*\left(M_0\right)\in(0,M_0), S_2^*\left(M_0\right)\in(0,M_0)$.

In this case we must have $\xi_1^*(M_0)=\xi_2^*(M_0)\geq\xi_3^*(M_0)$. In this case, we need to show neither S_1^* nor S_2^* drops. By Lemma 2, when one of S_i^* drops, the other must follow. Hence, suppose that both S_1^* and S_2^* drop, which implies Γ_0^* must strictly increase. ξ_3^* then strictly decrease, but at least ξ_2^* increases since $\theta_1+S_1^*\geq\theta_2+S_2^*$ from Lemma 2. Note $\xi_2^*(M_1)=\frac{r}{\theta_2+S_2^*(M_1)}-\frac{2r}{\theta_1+S_1^*(M_1)+\theta_2+S_2^*(M_1)}+\frac{1}{\theta_1+S_1^*(M_1)+\theta_2+S_2^*(M_1)}\geq\frac{r}{\theta_2+S_2^*(M_1)}-\frac{2r}{2[\theta_2+S_2^*(M_1)]}+\frac{1}{\theta_1+S_1^*(M_1)+\theta_2+S_2^*(M_1)}=\frac{1}{\theta_1+S_1^*(M_1)+\theta_2+S_2^*(M_1)}$. $\xi_2^*(M_0)=\frac{r}{\theta_2+S_2^*(M_0)}-\frac{2r}{\theta_1+S_1^*(M_0)+\theta_2+S_2^*(M_0)}+\frac{1}{\theta_1+S_1^*(M_0)+\theta_2+S_2^*(M_0)}=\frac{1}{\theta_1+S_1^*(M_0)+\theta_2+S_2^*(M_0)}$ since $\theta_1+S_1^*(M_0)=\theta_2+S_2^*(M_0)$. We thus have $\xi_2^*(M_1)>\xi_2^*(M_0)\geq\xi_3^*(M_0)>\xi_3^*(M_1)$. This means that the resources on $\Gamma_0^*(M_1)$ instead should be reallocated to S_2^*.

Case 3: $\Gamma_0^*\left(M_0\right)\in(0,M_0)$, $S_1^*\left(M_0\right)=0, S_2^*\left(M_0\right)\in(0,M_0)$.

In this case we must have $\xi_2^*(M_0) = \xi_3^*(M_0) \geq \xi_1^*(M_0)$. We need to show neither S_2^* nor Γ_0^* drops. Suppose that S_2^* drops, we must have $S_1^*(M_1) = 0$ by Lemma 2. This means Γ_0^* must increase. Hence, ξ_3^* decreases while ξ_2^* increases, which yields $\xi_2^* > \xi_3^*$. Thus, the resources on $\Gamma_0^*(M_1)$ instead should be reallocated to S_2^*.

Suppose Γ_0^* decreases, we must have the total resources on S_1^* and S_2^* increases. In this case, we must have S_2^* increases. Note $\xi_2^*(M_1) = \frac{r}{\theta_2+S_2^*(M_1)} - \frac{2r-1}{\theta_1+\theta_2+S_1^*(M_1)+S_2^*(M_1)}$. Consider two cases.

First, if $S_1^*(M_1) = 0$, then $\xi_2^*(M_1) = \frac{r}{\theta_2+S_2^*(M_1)} - \frac{2r-1}{\theta_1+\theta_2+S_2^*(M_1)}$, which implies $\xi_2^*(M_1) < \xi_2^*(M_0)$ as S_2^* has increased. Second, $S_1^*(M_1) > 0$. Because $\theta_1+S_1^*(M_1) = \theta_2+S_2^*(M_1)$, $\xi_2^*(M_1) = \frac{1}{2(\theta_2+S_2^*(M_1))}$. It also implies $\xi_2^*(M_1) < \xi_2^*(M_0)$ because $\xi_2^*(M_0) = \frac{r}{\theta_2+S_2^*(M_0)} - \frac{2r-1}{\theta_1+\theta_2+S_2^*(M_0)} = \frac{r}{\theta_2+S_2^*(M_0)} - \frac{2r}{\theta_1+\theta_2+S_2^*(M_0)} + \frac{1}{\theta_1+\theta_2+S_2^*(M_0)} > \frac{1}{\theta_1+\theta_2+S_2^*(M_0)} \geq \frac{1}{2(\theta_2+S_2^*(M_0))}$.

Thus eventually, ξ_3^* increases while ξ_2^* decreases. This means the resources on S_2^* instead should be reallocated to $\Gamma_0^*(M_1)$.

Case 4: $\Gamma_0^*(M_0) \in (0, M_0)$, $S_1^*(M_0) \in (0, M_0)$, $S_2^*(M_0) \in (0, M_0)$.

In this case we must have $\xi_1^*(M_0) = \xi_2^*(M_0) = \xi_3^*(M_0)$. We need to show none of the three choice variables can decrease. Combining the arguments in Cases 2 and 3 leads to this result.

Case 5: $\Gamma_0^*(M_0) = M_0$, $S_1^*(M_0) = 0$, $S_2^*(M_0) = 0$.

In this case we must have $\xi_3^*(M_0) \geq \max\{\xi_1^*(M_0), \xi_2^*(M_0)\}$. We need to show Γ_0^* cannot drop. Similar arguments as in Case 3 would apply.

Q.E.D. ■

Proof of Proposition 4

Proof. We first consider the case of severe asymmetry ($\theta_1 - \theta_2 > M$) where S_1^* must be zero. Clearly, the claim is true for the case of Proposition 1(a) and 1(b). For case 1(c), the equilibrium subsidy S_2^* is uniquely determined by the equation $\frac{r}{\theta_2+S_2} - \frac{2r-1}{\theta_1+(\theta_2+S_2)} = \frac{r}{V_0+M-S_2}$. When V_0 increases, RHS would decrease in response, which requires a larger S_2^* to rebalance the equation.

We then consider the case of mild asymmetry ($\theta_1 - \theta_2 \leq M$). A similar logic applies in equilibrium with only $S_2^* > 0$.

We then consider the equilibrium that involves positive subsidies to both firms. The equilibrium amounts of S_1^*, as well as $S_2^* = S_1^* + (\theta_1 - \theta_2)$ are uniquely determined by the equation $\frac{1}{2(\theta_1+S_1^*)} = \frac{r}{V_0+M-(\theta_1-\theta_2)-2S_1^*}$. When V_0 increases, RHS strictly decreases, which thus requires an increase in S_1^* to restore the balance.

An increasing V_0 could vary the type of the equilibrium. However, by the proofs of Proposition 1 and 2, S_1^* and S_2^* are continuous at thresholds. In addition, as pointed out in the text, an increasing V_0 increases the likelihood of the equilibrium with the entire budget to be allocated to subsidies. We then conclude that S_i^* increases with V_0.

Q.E.D. ■

Proof of Proposition 5

Proof. We consider the impact of a marginal increase in r on the equilibrium in two possible cases.

Case 1: $\theta_1 - \theta_2 > M$.

In this case, $S_1^* = 0$. We claim that in any equilibrium where $S_2^* > 0$, S_2^* strictly decreases with r. The equilibrium condition (17) can be rewritten as

$$\frac{\theta_1}{[\theta_1 + (\theta_2 + S_2^*)]\,(\theta_2 + S_2^*)} + \frac{1-r}{r[\theta_1 + (\theta_2 + S_2^*)]} = \frac{1}{(V_0 + M - S_2^*)}. \tag{29}$$

Assume there is an equilibrium with $S_2^* > 0, \Gamma_0^* > 0$. We fix this equilibrium and hold S_2^* constant. For an increase in r, the LHS of (29) would strictly decrease, while the RHS remains constant. To restore the balance, S_2 must be reduced.

Case 2: $\theta_1 - \theta_2 \leq M$.

Consider equation (28). Assume there is an equilibrium with $S_1^*, S_2^*, \Gamma_0^* > 0$ and we fix this equilibrium and hold S_1^* (as well as S_2^* since $S_2^* = S_1^* + (\theta_1 - \theta_2)$) constant. Imagine a marginal increase in r. We would see that the RHS of (28) would strictly increase, while RHS remains constant. To restore the balance, S_1^* (and S_2^* since $S_2^* = S_1^* + (\theta_1 - \theta_2)$) must be reduced.

By the same argument as we presented in case 1, this result holds in an equilibrium with $S_1^* = 0$ and $S_2^*, \Gamma_0^* > 0$.

In the reasoning we lay out above, we implicitly assume that the increase in r is marginal such that it does not cause a differing type of equilibrium. As aforementioned, an increase in r raises all the cutoffs for V_0 for different types of equilibria. Thus, the claim continues to hold when the change in r is not a marginal one. In case 1, assume that an increase in r causes $V_0 + M$ to fall below $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$, then S_2^* would drop to zero (Proposition 1). In case 2, the same would happen to S_1^* when an increase in r causes $V_0 + M$ falls below $2r\theta_1 + (\theta_1 - \theta_2)$ (Proposition 2). Such a change would cause S_2^* to decline as well. This can be seen by the proof laid out above and the fact that S_2^* is continuous on $V_0 + M$.

Q.E.D. ■

Proof of Lemma 4

Proof. Without loss of generality, we consider a case of $0 < \tilde{\theta}_3 \leq \tilde{\theta}_2 \leq \tilde{\theta}_1$. If $\frac{\tilde{\theta}_1\tilde{\theta}_2}{\tilde{\theta}_1+\tilde{\theta}_2} < \tilde{\theta}_3$, then $\frac{\tilde{\theta}_1\tilde{\theta}_3}{\tilde{\theta}_1+\tilde{\theta}_3} = \frac{\tilde{\theta}_1}{\mathbf{1}+(\tilde{\theta}_1/\tilde{\theta}_3)} \leq \frac{\tilde{\theta}_1}{\mathbf{1}+(\tilde{\theta}_1/\tilde{\theta}_2)} = \frac{\tilde{\theta}_1\tilde{\theta}_2}{\tilde{\theta}_1+\tilde{\theta}_2} < \tilde{\theta}_3 \leq \tilde{\theta}_2$.Thus $\frac{\tilde{\theta}_1\tilde{\theta}_3}{\tilde{\theta}_1+\tilde{\theta}_3} < \tilde{\theta}_2$. Moreover, $\frac{\tilde{\theta}_2\tilde{\theta}_3}{\tilde{\theta}_2+\tilde{\theta}_3} = \frac{\tilde{\theta}_3}{\mathbf{1}+(\tilde{\theta}_3/\tilde{\theta}_2)} \leq \frac{\tilde{\theta}_3}{\mathbf{1}+(\tilde{\theta}_3/\tilde{\theta}_1)} = \frac{\tilde{\theta}_1\tilde{\theta}_3}{\tilde{\theta}_1+\tilde{\theta}_3} < \tilde{\theta}_2 \leq \tilde{\theta}_1$.Thus (a) holds. If $\frac{\tilde{\theta}_1\tilde{\theta}_2}{\tilde{\theta}_1+\tilde{\theta}_2} \geq \tilde{\theta}_3$, then $\tilde{\theta}_1 > \tilde{\theta}_3$ and $\tilde{\theta}_2 > \tilde{\theta}_3$.We must have then (b) holds. ■

Proof of Proposition 6

Proof. (a) When $\tilde{\theta}_k > \frac{\prod_{j\neq k}\tilde{\theta}_j}{\sum_{j\neq k}\tilde{\theta}_j}$, Lemma 4(a) implies that $\tilde{\theta}_i > \frac{\prod_{j\neq i}\tilde{\theta}_j}{\sum_{j\neq i}\tilde{\theta}_j}, \forall i$.With this condition, clearly $\frac{\partial \pi_i}{\partial x_i}|_{x_i=0} = \frac{\tilde{\theta}_i}{\sum\limits_{j\neq i}\tilde{\theta}_j x_j}V - 1 > 0$ when the other two contestants play a Nash equilibrium assuming $x_i = 0$. Thus corner solution does not exist, i.e. an equilibrium with positive effort for every contestant must prevail. Denote $\tilde{Y} \equiv \tilde{\theta}_1 x_1 + \tilde{\theta}_2 x_2 + \tilde{\theta}_3 x_3$. The equilibrium condition can be written as $\tilde{Y} - \tilde{\theta}_i x_i = \frac{\tilde{Y}^2}{V\tilde{\theta}_i}, \forall i \in \{1,2,3\}$. Adding the three equations together yields $2\tilde{Y} = (\frac{\tilde{Y}^2}{\tilde{\theta}_1} + \frac{\tilde{Y}^2}{\tilde{\theta}_2} + \frac{\tilde{Y}^2}{\tilde{\theta}_3})\frac{1}{V}$. It follows that $\tilde{Y} = \frac{2V}{\frac{1}{\tilde{\theta}_1}+\frac{1}{\tilde{\theta}_2}+\frac{1}{\tilde{\theta}_3}} = \frac{2V\,\tilde{\theta}_1\,\tilde{\theta}_2\,\tilde{\theta}_3}{\tilde{\theta}_1\,\tilde{\theta}_2 + \tilde{\theta}_2\,\tilde{\theta}_3 + \tilde{\theta}_1\,\tilde{\theta}_3}$. Hence, we can solve for the equilibrium solutions by the fact $x_i = \frac{\tilde{Y}}{\tilde{\theta}_i} - \frac{1}{V}\left(\frac{\tilde{Y}}{\tilde{\theta}_i}\right)^2$.

(b) When $\tilde{\theta}_k \leq \frac{\prod_{j\neq k}\tilde{\theta}_j}{\sum_{j\neq k}\tilde{\theta}_j}$, an equilibrium with all three firms staying active does not exist as part (a) means $x_k < 0$. Lemma 4(b) gives $\tilde{\theta}_i > \frac{\prod_{j\neq i}\tilde{\theta}_j}{\sum_{j\neq i}\tilde{\theta}_j}, \forall i \neq k$. By the same arguments, there exists no equilibrium where either firm 1 or firm 2 staying inactive. Thus an equilibrium with an inactive k must prevail. It can be verify that with $x_j = \frac{\prod_{j\neq k}\tilde{\theta}_j}{(\sum_{j\neq k}\tilde{\theta}_j)^2}V, j \neq k$, $x_k = 0$ is the best response for k.

Q.E.D. ■

Proof of Proposition 7

Proof. Suppose at the optimal budget allocation, the subsidies are (S_1^*, S_2^*, S_3^*). If at this optimum all contestants are alive, then (25) implies that $S_1^* \leq S_2^* \leq S_3^*$, because $\frac{\partial\,\tilde{Y}}{\partial\,\tilde{\theta}_i}/V \gtreqless \frac{\partial\,\tilde{Y}}{\partial\,\tilde{\theta}_j}/V$ if and only if $\widetilde{\theta}_i \lesseqgtr \widetilde{\theta}_j$. In this case, it is standard to show $\widetilde{\theta}_1^* \geq \widetilde{\theta}_2^* \geq \widetilde{\theta}_3^*$.

If at this optimum, there is some contestant inactive, it must be true that firms 1 and 2 are active while firm 3 is inactive. The reasons are as follows. First, if less than two firms are active, then no positive effort is induced. This goes against the optimum. Thus it must be true that two firms are active and one is inactive. Second, it must be true that firms 1 and 2 are active while firm 3 is inactive. Without loss of generality, suppose at the optimum, firms 1and 3 are active. Putting all the subsidies on firms 1 and 3 must be optimal while it also maximizes the chance for keeping firm 2 inactive by Proposition 6(b). However, if we put the subsidy to firm 3 to firm 2, then it must be true that firm 3 is shut down while this reallocation increases $\tilde{Y}$.[26] Third, at the optimum the budget is allocated with active firms 1 and 2 as if firm 3 does not exist. For any given amount total subsidy, optimal allocation between the active two firms maximizes the product of their ex post efficiency. This also maximizes the chance for keeping firm 3 inactive by Proposition 6(b). In this case, the two-firm setting is restored, and the results of Lemma 2 apply. We thus have $\widetilde{\theta}_1^*$

[26]Note that $\theta_2 > \theta_3$ and $\frac{\widetilde{\theta}_1^*\,(\theta_2+S)}{\widetilde{\theta}_1^* + (\theta_2+S)} > \frac{\widetilde{\theta}_1^*\,(\theta_3+S)}{\widetilde{\theta}_1^* + (\theta_3+S)}, \forall S$. Thus $\theta_2 < \frac{\widetilde{\theta}_1^*\,(\theta_3+S)}{\widetilde{\theta}_1^* + (\theta_3+S)}$ implies $\theta_3 < \frac{\widetilde{\theta}_1^*\,(\theta_2+S)}{\widetilde{\theta}_1^* + (\theta_2+S)}$.

$\geq \widetilde{\theta}_2^* \geq \widetilde{\theta}_3^* = \theta_3$ and $S_2^* \geq S_1^* \geq S_3^* = 0$.

These results are summarized as follows: (1) The optimum does not exclude either firm 1 or firm 2; (2) $S_2^* \geq S_1^*$ while either $S_3^* \geq S_2^*$ or $S_3^* = 0$; (3) $\widetilde{\theta}_1^* \geq \widetilde{\theta}_2^* \geq \widetilde{\theta}_3^*$.

Q.E.D. ■

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