Leadership cycles*

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Abstract

We study a quality-ladder model of endogenous growth in which neither leaders nor outsiders are precluded from innovating. The model generates stochastic leadership cycles in which an incumbent can innovate several successive times, gradually increasing the magnitude of his technological lead before being replaced by a new entrant. Initially the incumbent is eager to enlarge his lead and does much of the research. However, if he is lucky enough to innovate repeatedly, after each successive innovation his profits increase. As a result, his propensity to invest in R&D decreases, until he is eventually overtaken with probability one and a new cycle starts.

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1 Introduction

We propose a simple quality-ladder model of endogenous growth in which neither leaders nor outsiders are precluded from innovating.[1] The model generates stochastic leadership cycles in which an incumbent can innovate several successive times, gradually increasing the magnitude of his technological lead before being replaced by a new entrant. Initially the incumbent is eager to enlarge his lead and does much of the research. If he is lucky enough to innovate repeatedly, however, after each successive innovation his profits increase. Therefore, his propensity to invest in R&D decreases until eventually he does no research and so is overtaken with probability one, and a new cycle starts. These leadership cycles endogenously create firms heterogeneity, with a skewed equilibrium firm size distribution that matches the findings of various empirical studies.[2] Over a leadership cycle, the expected rate of growth of an incumbent firm is inversely related to its size, a deviation from Gibrat's law that seems consistent with the empirical evidence.[3] While other models can explain some of these stylized facts,[4] leadership cycles seem a distinguishing feature of ours.

We generate these patterns by a minimal departure from first-generation endogenous growth models with quality ladders, such as Aghion and Howitt (1992), Segerstrom et al. (1990), or Grossman and Helpman (1991). A key

[1] Although technical change is frequently associated with the entry of new firms, there is ample empirical evidence that incumbents account for a sizeable share of the research done and often innovate repeatedly: see e.g. Malerba and Orsenigo (1995).

[2] Since in our model only incumbent firms are active at any point in time, the model can explain only the upper tail of the firm size distribution, which typically can be approximated quite well by a Pareto distribution: see e.g. Steindl (1965).

[3] See Hall (1987), Evans (1987) and Lotti et al. (2009).

[4] See, for instance, Kettle and Kortum (2004). The literature on innovations by leaders is discussed in more details below.

property of those models, and one which is at odd with reality, is that outsiders conduct all of the research so that the current technological leader is systematically replaced, implying also that firms are homogeneous and do not grow. This pattern of leapfrogging follows from Arrow's replacement effect when outsiders can conduct research on an equal footing with leaders. To allow for repeated innovations by leaders and the associated effects described above, we then posit that leaders are more efficient than outsiders in conducting the research.

This assumption is not new in the literature (see Barro and Sala-i-Martin (1994), Segerstrom and Zolnierek (1999), and Segerstrom (2007)) and seems natural in view of the cumulative nature of technical progress and the fact that innovative technological knowledge is often disclosed only partially. A consequence of these facts is that some knowledge useful to search for the next innovation may be available only to the latest inventor, making him more efficient in conducting research.[5]

However, the previous literature has overlooked some of the implications of this hypothesis, which become apparent only under appropriate ancillary assumptions. Barro and Sala-i-Martin (1994), for instance, assume that leaders have also a first-mover advantage, which leads to an equilibrium where they conduct all research preemptively.[6] By contrast, we retain the hypothesis that

[5] Incomplete disclosure is entailed, by definition, by trade secret protection. However, it may occur even if the innovation is patented, although in principle patenting requires full disclosure of the innovation. For, in practice knowledge may be difficult to codify and transmit to others, and firms seeking patent protection may have an incentive to disclose as little information as possible. As an extreme case, a firm may not be able to start searching for the $j+1$th innovation until after it has independently re-produced the jth one, as in step-by-step models of innovation (see e.g. Aghion et al. 2001).

[6] In fact, the first-mover advantage alone suffices to generate a pre-emption equilibrium in which incumbents conduct all research, as shown by Kettle and Griliches (2000), Denicolò (2001) and Etro (2004).

leaders and outsiders choose their R&D efforts simultaneously. This hypothesis is shared by most endogenous growth models, including Segerstrom and Zolnierek (1999) and Segerstrom (2007). However, these contributions assume that innovations are drastic, while we focus on the case of non-drastic innovations. With drastic innovations, the leader's technological advantage alone cannot produce an equilibrium in which leaders and outsiders simultaneously invest in R&D. To obtain such a result, Segerstrom and Zolnierek (1999) and Segerstrom (2007) further depart from first-generation models by assuming that there are decreasing returns to scale in research at the firm level, while we make the more standard assumption of constant returns. These different assumptions have an important consequence: in Segerstrom and Zolnierek (1999) and Segerstrom (2007) the probability that the leader is replaced is constant, while in our model it increases over his life cycle and is 100% after a certain number of successive innovations, which can be calculated explicitly and represents the maximum "length" of leadership cycles.[7]

Although our model is fully orthodox and makes no special assumptions apart from the leader's advantage in conducting research, it turns out that it may exhibit growth-rate indeterminacy. Many different levels of investment in research during different phases of a leadership cycle, that is to say, are consistent with the equilibrium for a range of parameter values.[8] This multiplicity of

[7] Another strand of the literature that has modelled repeated innovations by leaders posits that customers' loyalty guarantees to the leaders cheaper distributional channels, as in Stein (1997). These models generate an "entrenchment-of-monopoly" effect whereby the risk that the incumbent is replaced decreases with the duration of his leadership. In our model, by contrast, a leader who innovates repeatedly becomes fatter and fatter after every successive innovation, so the probability that he is overtaken increases over his life cycle.

[8] Although this indeterminacy is analogous to that uncovered by Cozzi (2007), there is one

equilibria is due to the assumption of constant returns to scale in research and may raise doubts about its appropriateness. However, not only does the constant returns assumption facilitate the comparison with the earlier literature, it seems also well grounded both empirically and theoretically. Surveying the empirical literature, Griliches (1990, p. 1677) notes that "in the major range of the data [...] there is little evidence for diminishing returns, at least in terms of patents per R&D dollar. That is not surprising, after all. If there were such diminishing returns, firms could split themselves into divisions or separate enterprises and escape them."[9] Accordingly, we retain the assumption of constant returns. However, to overcome the multiplicity of equilibria we assume that the equilibrium must be robust to the introduction of a negligible amount of decreasing returns into the R&D technology. This enables us to select a (generically) unique equilibrium, which is the one possessing the properties mentioned in the opening paragraph.

The rest of the paper is organized as follows: the next section outlines the model. Section 3 derives the conditions that must hold in a steady state equilibrium. Section 4 discusses the indeterminacy of the equilibrium and proposes a robustness criterion for selecting a unique equilibrium, called robust. Section 5 then characterizes the robust equilibrium. Section 6 summarizes and concludes. All proofs are collected in the Appendices.

important difference. In the models discussed by Cozzi, the indeterminacy of the equilibrium is created by the possibility of arbitrary (and yet fully legitimate) beliefs over out-of-the-equilibrium-path variables. In our model, by contrast, the indeterminacy arises even if all the relevant beliefs are fully determined.

[9] In time series comparisons, by contrast, it appears that the returns to R&D are significantly decreasing.

2 The model

For ease of comparison, we adapt the textbook quality-ladder model of Barro and Sala-i-Martin (2004).[10] However, our results are more general and can be reproduced in many other models, with or without scale effects.[11]

2.1 Preferences

The economy is populated by L identical, infinitely-lived individuals. Each individual inelastically supplies one unit of labour and has intertemporal preferences:

$$u = \int_0^\infty \left[\frac{c(t)^{1-\theta} - 1}{1-\theta}\right] e^{-\rho t} dt, \tag{1}$$

where $c(t)$ is consumption at time t, ρ is the rate of time preference, and $1/\theta$ is the intertemporal elasticity of substitution. Each individual maximizes (1) subject to the budget constraint:

$$c(t) + \dot{a}(t) = w(t) + r(t)a(t), \tag{2}$$

where $w(t)$ is the wage rate, $r(t)$ is the rate of interest, and $a(t)$ is the individual's wealth. Individuals are risk neutral, so in equilibrium by arbitrage all assets must yield the same instantaneous net rate of return $r(t)$.

2.2 Production

There is a unique final good in the economy that can be consumed, used to produce intermediate goods, or used in research. This good is taken as the

[10] With respect to Segerstrom and Zolnierek (1999), we make only two changes: first, we focus on the case of constant returns to scale in research at the firm level; second, we allow for non-drastic innovations.

[11] For example, scale effects could be eliminated using the approached used by Segerstrom (2007) in a model that shares several features with ours.

numeraire. It is produced in a perfectly competitive market using labour (which is in fixed supply) and a continuum of intermediate goods $\omega \in [0,1]$, the quality of which increases over time because of technical progress. We normalize the quality of all intermediate goods at time 0 to unity and denote by $\lambda > 1$ the size of each innovation. Thus, the quality of intermediate good ω of vintage $j(\omega,t)$ is $\lambda^{j(\omega,t)}$, where $j(\omega,t)$ denotes the number of innovations that have been achieved in industry ω by time t.

The final good can be produced according to the following constant-returns production function:

$$y(t) = \int_0^1 L^{1-\alpha} \left[\sum_{k=0}^{j(\omega,t)} \lambda^{j(\omega,t)-k} q(j-k,\omega,t) \right]^{\alpha} d\omega, \qquad 0 < \alpha < 1, \tag{3}$$

where L is labour input, $(1-\alpha)$ is the share of labour's income, and $q(j-k,\omega,t)$ denotes the input of the intermediate good of type ω and vintage $j-k$, so that $\sum_{k=0}^{j(\omega,t)} \lambda^{j(\omega,t)-k} q(j-k,\omega,t)$ is a quality-adjusted index of composite good ω that combines all past generations of intermediate goods of type ω. Assuming Bertrand competition, in equilibrium only the latest vintage of each intermediate good is employed in the production of the final good. Normalizing labour supply to one, the production function of the final good can then be re-written as:

$$y(t) = \int_0^1 \left[\lambda^{j(\omega,t)} q(j,\omega,t) \right]^{\alpha} d\omega. \tag{4}$$

Profit maximization by perfectly competitive firms in the final good sector implies the following demand for the last vintage of the intermediate good of type ω:

$$q(j,\omega,t) = \alpha^{\frac{1}{1-\alpha}} \lambda^{\frac{\alpha}{1-\alpha} j(\omega,t)} p(j,\omega,t)^{-\frac{1}{1-\alpha}}, \tag{5}$$

where $p(j, \omega, t)$ is its price. This demand function has a constant elasticity $\frac{1}{1-\alpha}$, and each innovation shifts demand up by a constant factor $g \equiv \lambda^{\frac{\alpha}{1-\alpha}} > 1$.

2.3 Intermediate products

Independently of its type ω and vintage j, each unit of the intermediate good can be produced using one unit of final good. Thus, all past and present innovators have the same unit cost – which, in terms of the numeraire, is one – but they offer vertically differentiated products. With Bertrand competition, in equilibrium only the current leader will be active. However, the equilibrium price depends on what technology is available to his most efficient rival (i.e., the penultimate innovator), who can supply the next most productive vintage.

Let i denote the number of consecutive innovations achieved by the current leader. In our model, i is determined endogenously, which marks a key difference from standard quality-ladder models where the current leader never invests in R&D and so i is always equal to one. In equilibrium, a firm leading by i steps, denoted by ℓ_i, prices at:

$$p = \min[\lambda^i, \frac{1}{\alpha}], \tag{6}$$

where $\frac{1}{\alpha}$ is the monopoly price,[12] driving its rivals out of the market.

Notice that the equilibrium price is independent of j, ω and t, but depends on i. More precisely, denoting by m the minimum number of consecutive innovations that allow a firm to engage in monopoly pricing, it follows from equation (6) that if $i \geq m$, the leader charges the monopoly price $\frac{1}{\alpha}$ and collects the

[12] Since the demand function for each intermediate good has constant elasticity $\frac{1}{1-\alpha}$ and the unit cost is one, the monopoly price is always $\frac{1}{\alpha}$.

monopoly profit:

$$\pi_m(j,\omega,t) = g^{j(\omega,t)}\pi_M, \tag{7}$$

where $\pi_M \equiv (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$. If instead $i < m$, the leader engages in limit pricing, $p = \lambda^i$, obtaining:

$$\pi_i(j,\omega,t) = g^{j(\omega,t)}\pi_i, \tag{8}$$

where $\pi_i \equiv (\lambda^i - 1)\alpha^{\frac{1}{1-\alpha}}\lambda^{-\frac{i}{1-\alpha}}$. In this region, the greater is the leader's technological advantage, the greater the price he charges in equilibrium, and hence the greater his profits.

The variable m is implicitly defined as a function of λ and α by the inequalities $\lambda^m \geq \frac{1}{\alpha} > \lambda^{m-1}$.[13] If $m = 1$, or $\lambda \geq \frac{1}{\alpha}$, innovations are drastic. If instead $\lambda < \frac{1}{\alpha}$, innovations are non-drastic, which implies $m \geq 2$. In particular, if $m = 2$, or $\sqrt{\frac{1}{\alpha}} \leq \lambda < \frac{1}{\alpha}$, it takes a two-step lead to engage in monopoly pricing: we call these innovations *quasi-drastic*.

Independently of the magnitude of his lead, the technological leader is always the sole active firm. Thus, in equilibrium only the latest vintage of each intermediate good is produced and employed in the production of the final good, as was claimed earlier. Hence, the total output of intermediate goods is $Q(t) = \int_0^1 q(j,\omega,t)d\omega$, which represents also the amount of the final good used in the production of intermediate goods.

[13] That is, m is the smallest integer greater than or equal to $-\frac{\log\alpha}{\log\lambda}$.

2.4 The R&D sector

In each industry ω there is a sequence of patent races. As soon as innovation j is achieved, a free-entry, simultaneous-move race starts to achieve innovation $j+1$. The current technological leader and a mass of outsiders can participate in this race by investing the final good in independent R&D projects. The arrival of the innovation follows a Poisson stochastic process with a hazard rate that depends on R&D investment. Leaders and outsiders make their R&D investment decision simultaneously and independently. All firms can adjust their R&D expenditures at any point in time, but with a Poisson discovery process, in a steady state they will choose a constant level of R&D expenditure until someone succeeds and the next race starts.

A generic firm $s = o, \ell_i$ (where o stands for outsiders and ℓ_i for the current leader) that invests $R_s(j, \omega, t)$ units of the final good in R&D in sector ω in order to achieve innovation j will succeed with an instantaneous probability:

$$x_s(j, \omega, t) = \frac{R_s(j, \omega, t)}{c_s g^{j-1}}. \tag{9}$$

The parameters c_{ℓ_i} and c_o measure the R&D productivity of leaders and outsiders, respectively. All outsiders, including past innovators, have the same R&D productivity, but the leader is more productive: $c_o > c_{\ell_i}$. For simplicity, we assume that a leader's productivity in R&D is independent of the size of his lead: $c_{\ell_i} = c_\ell$.

Since R&D projects are independent, the aggregate instantaneous probability of success equals the sum of the individual probabilities. The aggregate hazard rate will thus be:

$$X(j,\omega,t) = x_{\ell_i}(j,\omega,t) + X_O(j,\omega,t) = \frac{R_{\ell_i}(j,\omega,t)}{c_\ell g^{j-1}} + \frac{\sum R_o(j,\omega,t)}{c_o g^{j-1}}, \qquad (10)$$

where X_O denotes the aggregate R&D effort of outsiders. The term g^{j-1} in the denominator of (10) means that research becomes increasingly difficult as new innovations arrive, an assumption that serves to guarantee the existence of a steady state.[14]

Denoting $R(j,\omega,t) \equiv R_{\ell_i}(j,\omega,t) + \sum R_o(j,\omega,t)$, the total amount of final good used in research is then $R(t) = \int_0^1 R(j,\omega,t)d\omega$. This completes the description of the model.

2.5 Steady state

As new innovations arrive, the productivity of intermediate goods increases, and hence the output of the final good increases. The productivity of any one intermediate good jumps up discretely at random intervals, but since there is a continuum of intermediate goods, by the law of large numbers, the economy can grow smoothly.

Abstracting from any transitional dynamics, in this paper we consider only the steady state. In a steady state the output of the final good, consumption, aggregate R&D expenditure, the aggregate output of intermediate goods, and the wage rate all grow at a constant rate, denoted by γ. Moreover, the fraction of industries in which the incumbent leads by i steps, denoted by κ_i, is constant,

[14] In a steady state, the expected waiting time to discovery, $1/X$, must be constant. Since R&D investment R grows at rate g from one race to the next, then in order for the aggregate hazard rate X to be constant the productivity of R&D must decline at rate g. This requires the knife-edge assumption that the ratio $R(j,\omega)/X(j,\omega)$ increases at rate g, which is standard in quality-ladder endogenous growth models (see e.g. Barro and Sala-i-Martin, 2004).

and the expected waiting time for innovations can depend only on i. This requires that[15]

$$x_{\ell_i}(j,\omega,t) = x_{\ell_i} \tag{11}$$

and

$$X_{O_i}(j,\omega,t) = X_{O_i}. \tag{12}$$

Averaging across industries, the expected waiting time for innovations is constant in a steady state.

3 Equilibrium conditions

This section derives the conditions that must hold in a steady state equilibrium. These conditions are summarized by two relationships that involve the interest rate r and the average aggregate hazard rate $X \equiv \sum_{i=1}^{m} \kappa_i X_i$. Before deriving these relationships, we calculate the steady state rate of growth, γ.

3.1 The growth rate

In a steady state, by definition, the κ_i's must be constant. Hence, the following conditions must hold:

$$\begin{aligned} \dot{\kappa}_1 &= \kappa_1 X_{O_1} + \kappa_2 X_{O_2} + ... + \kappa_m X_{O_m} - \kappa_1 X_1 = 0 \\ \dot{\kappa}_2 &= \kappa_1 x_{\ell_1} - \kappa_2 X_2 = 0 \\ & ... \\ \dot{\kappa}_m &= \kappa_{m-1} x_{\ell_{m-1}} - \kappa_m X_m = 0 \end{aligned} \tag{13}$$

[15] To avoid other possible sources of indeterminacy, we assume that the aggregate hazard rate must be the same in all industries in a steady state. See Cozzi (2007) for a discussion of this assumption and the consequences of relaxing it.

This system provides only $m-1$ independent equations, as the first equation can be obtained from the others. Together with the adding-up condition $\sum_{i=1}^{m} \kappa_i = 1$, these equations can be solved to get:

$$\kappa_1 = \left[1 + \sum_{i=2}^{m} \left(\frac{\prod_{j=1}^{i-1} x_{\ell_j}}{\prod_{j=2}^{i} X_j} \right) \right]^{-1} \tag{14}$$

and

$$\kappa_i = \frac{\prod_{j=1}^{i-1} x_{\ell_j}}{\prod_{j=2}^{i} X_j} \kappa_1 \qquad i = 2, ..., m. \tag{15}$$

Now insert (5) into (4) to obtain:[16]

$$y(t) = \alpha^{\frac{\alpha}{1-\alpha}} \bar{p}^{-\frac{\alpha}{1-\alpha}} \Lambda(t) \tag{16}$$

where $\Lambda(t) \equiv \int_0^1 \lambda^{\frac{\alpha}{1-\alpha} j(\omega,t)} d\omega$ is an intermediate good aggregate quality index

[16] Equation (16) is obtained as follows. Partition the set of industries $[0, 1]$ into m sub-sets $\varkappa_i$ (with $i = 1, 2, ..., m$) where the leader leads by 1, 2, ..., and m or more steps, respectively. Let κ_i denote the measure of $\varkappa_i$, i.e., the fraction of industries in which the leader leads by i steps. In a steady state κ_i is constant, although the set $\varkappa_i$ changes continuously. Since the equilibrium price depends only on i, we can reformulate $y(t)$ as follows:

$$\begin{aligned} y(t) &= \int_0^1 \left[\lambda^{j(\omega,t)} \alpha^{\frac{1}{1-\alpha}} \lambda^{\frac{\alpha}{1-\alpha} j(\omega,t)} p(j, \omega, t)^{-\frac{1}{1-\alpha}} \right]^{\alpha} d\omega \\ &= \alpha^{\frac{\alpha}{1-\alpha}} \sum_{i=1}^{m} \left[p_i^{-\frac{\alpha}{1-\alpha}} \int_{\varkappa_i} \lambda^{\frac{\alpha}{1-\alpha} j(\omega,t)} d\omega \right]. \end{aligned}$$

Since the probability that the leader has an i-step advantage is the same across industries, the variable $\lambda^{\frac{\alpha}{1-\alpha} j(\omega,t)}$ will be identically distributed over any subset $\varkappa_i$. Hence:

$$\int_{\varkappa_i} \lambda^{\frac{\alpha}{1-\alpha} j(\omega,t)} d\omega = \kappa_i \int_0^1 \lambda^{\frac{\alpha}{1-\alpha} j(\omega,t)} d\omega.$$

Substituting into the preceding expression we get:

$$y(t) = \alpha^{\frac{\alpha}{1-\alpha}} \left[\int_0^1 \lambda^{\frac{\alpha}{1-\alpha} j(\omega,t)} d\omega \right] \sum_{i=1}^{m} \kappa_i p_i^{-\frac{\alpha}{1-\alpha}}$$

whence equation (16) immediately follows.

that increases over time with technical progress, and $\bar{p} \equiv \left(\sum_{i=1}^{m} \kappa_i p_i^{-\frac{\alpha}{1-\alpha}}\right)^{-\frac{1-\alpha}{\alpha}}$ is an intermediate goods price index. Since $\bar{p}$ is constant in a steady state, equation (16) implies that the rate of growth of output must be the same as the rate of growth of the average quality of the intermediate goods, $\Lambda(t)$.

Since in an industry where the leader has an i-step advantage $j(\omega,t)$ jumps up to the next higher integer with a constant instantaneous probability X_i, the time derivative of $\Lambda(t)$ is:[17]

$$\begin{aligned} \dot{\Lambda}(t) &= \int_0^1 \left\{ \lambda^{\frac{\alpha}{1-\alpha}[j(\omega,t)+1]} - \lambda^{\frac{\alpha}{1-\alpha}j(\omega,t)} \right\} X(j+1,\omega,t) d\omega \\ &= (g-1)X\Lambda. \end{aligned} \tag{17}$$

It follows that the economy's rate of growth is

$$\gamma = (g-1)X. \tag{18}$$

3.2 The Euler equation

Now we turn to the equilibrium conditions. First, maximization of (1) under the budget constraint (2) leads to the familiar Euler condition:

$$\frac{\dot{c}(t)}{c(t)} = \frac{r-\rho}{\theta}. \tag{19}$$

For consumption to grow at a constant rate $\gamma = (g-1)X$, the interest rate r must be

[17] To derive the second line of equation (17) we proceed as in footnote 16 above. After partitioning the interval $[0,1]$ into m subsets $\varkappa_i$ and noting that the variable $\lambda^{\frac{\alpha}{1-\alpha}j(\omega,t)}$ is identically distributed over any subset $\varkappa_i$, we can rewrite the right-hand side of the first line of (17) as

$$\sum_{i=1}^{m} \left[(g-1)X_i \int_{\varkappa_i} \lambda^{\frac{\alpha}{1-\alpha}j(\omega,t)} d\omega \right] = \sum_{i=1}^{m} \left[(g-1)\kappa_i X_i \int_0^1 \lambda^{\frac{\alpha}{1-\alpha}j(\omega,t)} d\omega \right]$$

whence the second line of (17) immediately follows.

$$r = \theta(g-1)X + \rho. \tag{20}$$

This provides an increasing relationship between X and r.

3.3 The patent race equilibrium

The other equilibrium relationship is obtained from the analysis of firms' behavior in patent races.

3.3.1 The value of innovations

We start by determining the value of innovations. Let $V_1(j, \omega, t)$ be the value of leading by one step in industry ω at time t if j innovations have already been made in that industry. This is given by the following Bellman equation (to simplify the notation, we suppress the indices ω and t when there is no risk of confusion):

$$rV_1(j) = \max_{x_{\ell_1}} \left[\pi_1(j) - X_1 V_1(j) + x_{\ell_1} V_2(j+1) - c_\ell g^j x_{\ell_1} \right], \tag{21}$$

where $V_2(j+1)$ is the value of leading by two steps if $j+1$ innovations have been previously achieved. The interpretation of equation (21) is simple. The right-hand side is the flow value of leading by one step. A one-step leader earns the flow profit $\pi_1(j)$ and incurs the flow cost $c_\ell g^j x_{\ell_1}$ until innovation $j+1$ arrives. When innovation $j+1$ is achieved, which occurs with an instantaneous aggregate probability X_1, the leader incurs a capital loss $V_1(j)$, but in case he himself succeeds, an event whose probability is x_{ℓ_1}, he obtains $V_2(j+1)$. That is, $V_2(j+1) - V_1(j)$ is the net capital gain obtained by a one-step leader who innovates again. The leader chooses x_{ℓ_1} to maximize his present discounted profits. Equation (21) states that such maximized profits must guarantee a rate of return on the leader's asset, $V_1(j)$, equal to the equilibrium interest rate r.

The value of leading by two steps, $V_2(j+1)$, is in turn determined by the following Bellman equation

$$rV_2(j+1) = \max_{x_{\ell_2}} \left[\pi_2(j+1) - X_2 V_2(j+1) + x_{\ell_2} V_3(j+2) - c_\ell g^{j+1} x_{\ell_2} \right], \quad (22)$$

where $V_3(j+2)$ is the value of leading by three steps, and so on. After m successive innovations, the leader becomes an unconstrained monopolist in the product market. This implies that

$$V_{m+1}(j+m) = V_m(j+m), \quad (23)$$

since a firm leading by m or more steps will price at $p_M = \frac{1}{\alpha}$ and earn monopoly profits, irrespective of the magnitude of its lead.

Recall that in a steady state the profit earned by a firm leading by i steps increases by a constant factor g from one period to the next, i.e.,

$$\pi_i(j, \omega, t) = g^j \pi_i. \quad (24)$$

This property is evidently inherited by the value functions, so we must have

$$V_i(j) = g^j V_i, \quad (25)$$

where $V_i = V_i(0)$. Then, we can rewrite the Bellman equations as follows:

$$\begin{aligned} rV_1 &= \max_{x_{\ell_1}} \left[\pi_1 - X_1 V_1 + g x_{\ell_1} V_2 - c_\ell x_{\ell_1}\right], \\ rV_2 &= \max_{x_{\ell_2}} \left[\pi_2 - X_2 V_2 + g x_{\ell_2} V_3 - c_\ell x_{\ell_2}\right], \\ &\quad ... \\ rV_m &= \max_{x_{\ell_m}} \left[\pi_M - X_m V_m + g x_{\ell_m} V_m - c_\ell x_{\ell_m}\right]. \end{aligned} \tag{26}$$

3.3.2 The free-entry condition

Consider next a generic outsider who participates in a patent race in an industry where the leader leads by i steps. If he wins, he obtains a one-step leadership, the value of which is V_1. Thus, the expected discounted profit of any individual outsider who invests $c_o g^{j-1} x_{o_i}$ units of the final good to obtain innovation j is

$$\frac{x_{o_i} V_1(j) - c_o g^{j-1} x_{o_i}}{r + X_i}.$$

By the free entry condition, this cannot be strictly positive:

$$gV_1 - c_o \leq 0, \tag{27}$$

with $x_{o_i} = 0$ (and hence $X_{O_i} \equiv \sum x_{o_i} = 0$) if the inequality is strict.

3.3.3 Equilibrium

For any given interest rate r, an equilibrium in the patent race is a list of variables $(X_1, X_2, ..., X_m, x_{\ell_1}, x_{\ell_2}, ..., x_{\ell_m}, V_1, V_2, ..., V_m)$ that satisfy the Bellman equations (26), the free entry condition (27), and the complementary slackness condition that $X_{O_i} = X_i - x_{\ell_i}$ can be positive only if $gV_1 = c_o$.

Given r, a patent race equilibrium uniquely determines the κ_i's and hence the aggregate hazard rate X. By varying r, one can then construct a (non increasing) relationship between r and X that together with (20) determines the model's equilibrium r^* and X^*.

As we shall see, however, for any given r there may be multiple patent race equilibria. In this case, the model's equilibrium is not unique.

3.4 The goods market equilibrium

Proprietary technological knowledge is the only asset in our model economy, and it yields a return, $ra(t)$, which consists of the extra-profits earned by firms holding market power. Aggregating across firms, this equals

$$ra(t) = \alpha y(t) - Q(t). \tag{28}$$

The increase in the net value of the asset must in turn equal aggregate R&D investment:

$$\dot{a}(t) = R(t). \tag{29}$$

Plugging (27) and (28) into the budget constraint (2), one sees that if the labour market clears ensuring that $w(t) = (1-\alpha)y(t)$, then the goods market equilibrium condition

$$y(t) = c(t) + Q(t) + R(t) \tag{30}$$

is automatically satisfied at any point in time (Walras' law).

4 Indeterminacy and equilibrium selection

Now we use the equilibrium conditions derived in the previous section to solve for the model's steady state equilibria. We show that the model admits infinitely many equilibria for a set of parameter values of positive measure and we propose a criterion for selecting a unique equilibrium.

4.1 Preliminary results

We first provide a convenient characterization of the patent race equilibrium in terms of a set of inequalities and complementary slackness conditions.

Lemma 1 *For any given interest rate r, a list of non-negative variables (X_1, X_2,..., $X_m, x_{\ell_1}, x_{\ell_2}, ..., x_{\ell_m}, V_1, V_2, ..., V_m$) such that $X_{O_i} = X_i - x_{\ell_i} \geq 0$ for $i = 1, ..., m$ is a patent-race equilibrium if and only if they satisfy the following inequalities, with the associated complementary slackness conditions :*

$$gV_{i+1} - V_i - c_\ell \leq 0 \text{ and } x_{\ell_i}(gV_{i+1} - V_i - c_\ell) = 0 \ \forall i = 1, ..., m-1 \quad (31)$$

$$(g-1)V_m - c_\ell \leq 0 \text{ and } x_{\ell_m}[(g-1)V_m - c_\ell] = 0 \quad (32)$$

$$gV_1 - c_o \leq 0 \text{ and } \left(\sum_{i=1}^{m} X_{O_i}\right)(gV_1 - c_o) = 0 \quad (33)$$

Moreover, in any equilibrium

$$V_i = \frac{\pi_i}{r + X_{O_i}} \qquad \forall i = 1, 2, ..., m. \quad (34)$$

Using this characterization of the patent race equilibrium, it is easy to prove that if a leader's advantage in conducting the research is very large, the leader can behave as if he were unconstrained by outside competition and hence does all of the research.

Lemma 2 *If $\frac{c_o}{c_\ell} > \frac{g}{g-1}$, there is a unique steady state equilibrium; in this equilibrium, outsiders do not invest in R&D.*

However, uniqueness is not guaranteed for $\frac{c_o}{c_\ell} \leq \frac{g}{g-1}$; in this interval, to the contrary, the equilibrium is indeterminate. This indeterminacy arises because the outsiders' R&D investment is uniquely determined for any given r, but no equilibrium condition pins down the leaders' R&D investments.

Before proceeding, we state two more preliminary results. The next Lemma generalizes a result originally obtained by Segerstrom and Zolnierek (1999), who showed that when $m = 1$ there is no equilibrium in which the leader and outsiders simultaneously invest in R&D.

Lemma 3 *If $\frac{c_o}{c_\ell} \neq \frac{g}{g-1}$, there is no equilibrium in which $\sum_{i=1}^{m} X_{O_i} > 0$ and $x_{\ell_i} > 0$ for all $i = 1, 2, ..., m$; in other words, there is no equilibrium in which outsiders and all leaders simultaneously invest in R&D.*

Lemma 4 provides a partial converse to Lemma 2.

Lemma 4 *If $\frac{c_o}{c_\ell} < \frac{g}{g-1}$, there is no equilibrium with positive growth in which outsiders do not invest in R&D.*

4.2 Indeterminacy

Now we are ready to illustrate the source of indeterminacy. For ease of exposition, we focus on the case of quasi-drastic innovations ($m = 2$) and assume that $\frac{c_o}{c_\ell} < \frac{g}{g-1}$. Lemma 4 then implies that outsiders must be willing to invest in R&D, and Lemma 3 that a leader who leads by two steps cannot invest. The first condition implies $V_1 = \frac{c_o}{g}$ (from the outsiders' free-entry condition), the second $V_2 = \frac{\pi_M}{X_2 + r}$ (from ℓ_2's Bellman equation with $x_{\ell_2} = 0$). Hence, there are only two possible cases: either a leapfrogging equilibrium where $x_{\ell_1} = 0$, or an equilibrium in which ℓ_1 invests in R&D along with outsiders, i.e., $x_{\ell_1} > 0$. For our purposes, it suffices to focus on the latter equilibria, in which a leader who leads by one step must be indifferent between investing in R&D or not. This implies $gV_2 - V_1 - c_\ell = 0$ and $V_1 = \frac{\pi_1}{X_{O_1}+r}$ (from ℓ_1's Bellman equation). Together with the two conditions dereived above and the Euler equation (20), this leaves us with five equations in six unknowns: $r, V_1, V_2, X_2, X_{O_1}$ and x_{ℓ_1}.[18] That is, we are one equation down. In particular, V_1 and V_2 are uniquely determined as $\frac{c_o}{g}(\equiv \tilde{V}_1)$ and $\frac{c_o+gc_\ell}{g^2}(\equiv \tilde{V}_2)$, respectively; for any given r, X_2 and X_{O_1} are then uniquely determined as $X_2 = \frac{\pi_M}{\tilde{V}_2} - r$ and $X_{O_1} = \frac{\pi_1}{\tilde{V}_1} - r$, but no condition is left to pin down x_{ℓ_1}.

Since at equilibrium ℓ_1 is indifferent between investing in R&D or not, x_{ℓ_1} can then take any non-negative value which is consistent with the other equilibrium conditions. For example, $x_{\ell_1} = 0$ is always an equilibrium. In this case, $\kappa_1 = 1$ and hence $X = X_{O_1}$, so one obtains the familiar leapfrogging equilibrium. As x_{ℓ_1} increases, $X = \kappa_1 X_1 + \kappa_2 X_2$ increases and hence so does r. Two cases are then possible. Either there exists a finite value of x_{ℓ_1}, say $x_{\ell_1}^+$, such that $r^* = \frac{\pi_1}{\tilde{V}_1}$, in which case x_{ℓ_1} cannot grow larger than $x_{\ell_1}^+$ for otherwise the equilibrium condition $V_1 = \frac{\pi_1}{X_{O_1}+r}$ could not hold, or no such finite value exists.[19] In the former case, x_{ℓ_1} can take on any value in the interval $[0, x_{\ell_1}^+]$, and, accordingly, X ranges from $\underline{X}$ to $X^+ = \frac{\frac{\pi_1}{\tilde{V}_1} - \rho}{\theta(g-1)}$. In the latter case, x_{ℓ_1} can grow unbounded.

[18] Given x_{ℓ_1}, X_{O_1} and X_2, one can easily calculate κ_1 and κ_2 and hence X. That is, one can re-express equation (20) in terms of r, x_{ℓ_1}, X_{O_1} and X_2 only.

[19] Notice that when $r^* = \frac{\pi_1}{\tilde{V}_1}$, $X_2 = \frac{\pi_M}{\tilde{V}_2} - r$ is still non-negative since $\frac{\pi_M}{\tilde{V}_2} \geq \frac{\pi_1}{\tilde{V}_1}$ when $\frac{g\pi_1}{g\pi_M - \pi_1} \leq \frac{c_o}{c_\ell} < \frac{g}{g-1}$.

It can be shown that X increases monotonically with x_{ℓ_1} and[20] $\lim_{x_{\ell_1}\to\infty} X = 2\left(\frac{\pi_M}{\tilde{V}_2} - r\right)$. (The reason why X stays finite in the limit is that κ_1 tends to zero and κ_2 tends to 1, and X_2 is finite).

A similar logic applies to the case $m > 2$, except that now several variables (i.e., all the x_{ℓ_i}'s) can be indeterminate. A noteworthy property of these equilibria is that while the leaders's investments in R&D are indeterminate, the outsiders' aggregate R&D effort is fully determined for any given r. This means that greater investment in R&D by leaders does not crowd out investment by outsiders directly. There is only the indirect crowding out caused by the associated increase in the equilibrium interest rate (a general equilibrium effect).

4.3 Robust equilibria

The multiplicity of equilibria pointed out abve is due to our assumption of constant returns to scale in research. Although this assumption is standard in the endogenous growth literature and seems supported by the empirical evidence, it has the unpalatable consequence that at equilibrium firms are indifferent as to the amount of research they do. In models in which only the outsiders (respectively, only the leaders) conduct the research, the level of investment in research is fully determined by the condition that outsiders (respectively, leaders) must be indifferent between investing in R&D or not. But when both leaders and outsiders simultaneously invest in R&D, these conditions do not pin down a

[20] Notice that

$$X = \frac{\frac{\pi_M}{\tilde{V}_2} - r}{\frac{\pi_M}{\tilde{V}_2} - r + x_{\ell_1}} \left(\frac{\pi_1}{\tilde{V}_1} - r + x_{\ell_1}\right) + \frac{x_{\ell_1}}{\frac{\pi_M}{\tilde{V}_2} - r + x_{\ell_1}} \left(\frac{\pi_M}{\tilde{V}_2} - r\right)$$

unique equilibrium.

One way of selecting a unique equilibrium would be to confine one's attention to "super-stationary" equilibria in which the expected waiting time for innovations is constant not only in the aggregate but also within each industry. That is, in a super-stationary equilibrium the aggregate R&D effort would not depend on the number of consecutive innovations achieved by the leader:

$$X_i = x_{\ell_i} + X_{O_i} = X \qquad \text{for all } i = 1, 2, ..., m. \tag{35}$$

Although super-stationary equilibria seem appealing, they do not always exist.[21] Therefore, we follow a different approach. Our contention is that reasonable equilibria should be robust to perturbations that introduce a small degree of decreasing returns into the R&D technology. More formally, we define a "robust" equilibrium as the limit of the equilibria that arise with decreasing returns to scale in R&D as the degree of decreasing returns becomes negligible.

Now we show that this stability criterion selects an unique equilibrium (except for the special case $\frac{c_o}{c_\ell} = \frac{g}{g-1}$); in particular, it selects the super-stationary equilibrium if one exists. One objective of our procedure for selecting an equilibrium is to make sure that the property that both leaders and outsiders simultaneously invest in R&D is not an artifact of the assumption of decreasing returns in R&D. To this end, we assume that decreasing returns operate at the aggregate level rather than at the firm level, and we take the limit as the degree of decreasing returns tends to zero. Here we just sketch our argument

[21] To be sure, the leapfrogging equilibrium always exists (set $x_{\ell_i} = 0$ for all i) and is super-stationary in a degenerate sense.

informally, leaving the details to Appendix B.

Assume that the instantaneous probability that a generic firm $s = o, \ell_i$ (which is active in an industry in which the leader leads by i steps) succeeds is $\frac{x_s}{X_i} X_i^{\gamma}$, where $0 < \gamma \leq 1$ is a parameter that captures the returns to scale in R&D. The aggregate probability of success is X_i^{γ}. The case of constant returns corresponds to $\gamma = 1$.

With this new R&D technology, the free-entry condition by outsiders requires that

$$\frac{\frac{x_{o_i}}{X_i} X_i^{\gamma} V_1(j) - c_o g^{j-1} x_{o_i}}{r + X_i}$$

is non-positive and is zero if outsiders invest in R&D. Thus, if outsiders invest in R&D the following condition must hold:

$$X_i^{\gamma - 1} = \frac{c_o}{g V_1}. \tag{36}$$

Since V_1 is constant, X_i must be independent of i as long as outsiders invest in R&D. However, X_i can depend on i if outsiders do not invest. These properties hold for any $\gamma < 1$, and hence they must hold also in the equilibrium that is the limit of the decreasing returns equilibria as $\gamma \to 1$.

Summarizing, in a robust equilibrium the aggregate R&D investment X_i is independent of i as long as outsiders invest in R&D. Appendix B proves the following:

Lemma 5 *If $\frac{c_o}{c_\ell} \neq \frac{g}{g-1}$, there exists a unique robust equilibrium. The robust equilibrium coincides with the super-stationary equilibrium if one exists.*

5 Leadership cycles

In this section we analyze the properties of robust equilibria. These properties crucially depend on the relative R&D productivity of leaders and outsiders, $\frac{c_o}{c_\ell} > 1$. When the leader's advantage in conducting the research is small (i.e., $\frac{c_o}{c_\ell}$ is close to one), the standard leapfrogging equilibrium arises in which the current leader is systematically replaced by outsiders. When instead the leader's advantage is sufficiently large (to be precise, the condition is $\frac{c_o}{c_\ell} > \frac{g}{g-1}$), a persistent leadership equilibrium emerges in which leaders conduct all of the research and are unconstrained by outside competition. The most interesting case arises for intermediate values of $\frac{c_o}{c_\ell}$. There exists an interval, which is non degenerate when innovations are non drastic, in which both leaders and outsiders simultaneously invest in R&D.

5.1 Benchmarks

To proceed, we briefly review the leapfrogging and persistent leadership equilibria, which are familiar from the earlier literature (see e.g. Barro and Sala-i-Martin, 2004), and we identify the set of parameter values where our model generates those equilibria.

5.1.1 *Leapfrogging*

If leaders do not invest in R&D, we can set $x_{\ell_1} = 0$ in (21) obtaining:

$$V_1 = \frac{\pi_1}{r + X}. \tag{37}$$

Inserting this equation into the free-entry condition, which now necessarily holds as an equality, gives $X = \frac{g\pi_1}{c_o} - r$. This provides a decreasing relationship between X and r that, jointly with the Euler equation (20), determines the equilibrium hazard rate under leapfrogging:

$$\underline{X} = \frac{\frac{g\pi_1}{c_o} - \rho}{1 + \theta(g-1)}.\qquad (38)$$

[22]

5.1.2 *Persistent leadership*

If only the leader invests in R&D, eventually he will gain a lead large enough to engage in monopoly pricing. The value of holding such unconstrained leadership is:[23]

$$V = \frac{\pi_M}{r}.\qquad (39)$$

[22] This is positive only if $\frac{g\pi_1}{c_o} > \rho$. If this inequality is reversed, the economy stagnates indefinitely.

[23] To derive equation (39), notice that the relevant Bellman equation for such unchallenged leader is

$$rV(j) = \max_{x_\ell} \left[g^j \pi_M - x_\ell V(j) + x_\ell V(j+1) - c_\ell g^j x_\ell \right].$$

This equation says that securities issued by the leader pay the flow profit $g^j \pi_M$ until the next innovation arrives, plus the capital gain $x_\ell [V(j+1) - V(j)]$ that will be obtained when the next innovation occurs, less the R&D expenditure $c_\ell g^j x_\ell$. In a steady state, $V(j) = g^j V$, and hence the equation above can be rewritten as

$$rV = \max_{x_\ell} \left[\pi_M + x_\ell (g-1)V - c_\ell x_\ell \right]$$

Inspection of the right-hand side of this expression reveals that if $c_\ell < (g-1)V$, then V increases with x_ℓ unboundedly. This is inconsistent with equilibrium, so in equilibrium $c_\ell \geq (g-1)V$ must hold. If the inequality is strict, the optimal choice is $x_\ell = 0$; if instead $c_\ell = (g-1)V$, then any non-negative value of x_ℓ is optimal. In both cases,

$$\max_{x_\ell} \left[\pi_M + x_\ell (g-1)V - c_\ell x_\ell \right] = \pi_M$$

and hence equation (39) follows.

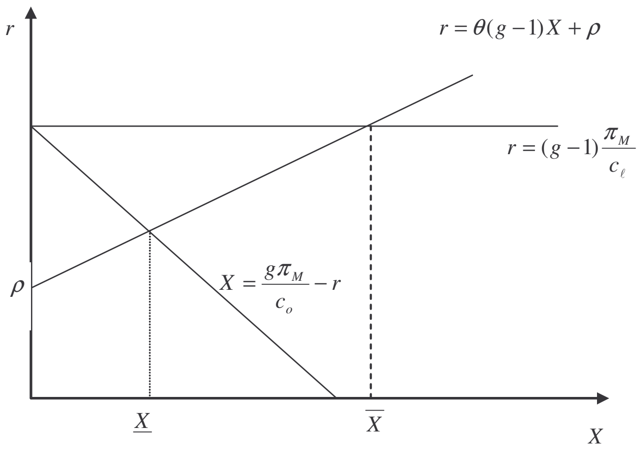

Figure 1: The equilibrium with leapfrogging and persistent leadership withdrastic innovations and $\frac{c_o}{c_\ell} = \frac{g}{g-1}$.

Since at any equilibrium with positive growth one must have $c_\ell = (g-1)V$ (see footnote 23), it follows that $r = (g-1)\frac{\pi_M}{c_\ell}$. Substituting into (20) one gets:

$$\overline{X} = \frac{\pi_M}{\theta c_\ell} - \frac{\rho}{\theta(g-1)}.^{24} \tag{40}$$

Figure 1 depicts the leapfrogging and persistent leadership equilibria in the special case of drastic innovations and $\frac{c_o}{c_\ell} = \frac{g}{g-1}$.

The next Proposition summarizes cases in which our model generates equilibria that have already been studied in the previous literature. To state this result more conveniently, let $F(g)$ be defined as $\frac{g}{g-1}$ when innovations are drastic ($m = 1$), $\frac{g\pi_1}{g\pi_M - \pi_1}$ when innovations are quasi-drastic ($m = 2$), and $\frac{g\pi_1}{g\pi_2 - \pi_1}$ in all other cases ($m \geq 3$).

Proposition 6 *i) If* $\frac{c_o}{c_\ell} > \frac{g}{g-1}$, *there is a unique equilibrium in which outsiders*

[24] This is positive only if $\frac{(g-1)\pi_M}{c_\ell} > \rho$. If this inequality is reversed, the economy stagnates indefinitely.

do not invest in R&D and hence $X = \overline{X}$ *(persistent leadership).*

ii) If $\frac{c_o}{c_\ell} < F(g)$*, there is a unique equilibrium in which outsiders do all of the research and hence* $X = \underline{X}$ *(leapfrogging).*

iv) If $\frac{c_o}{c_\ell} = \frac{g}{g-1}$*, there exist infinitely many equilibria in which both outsiders and all leaders invest in R&D simultaneously;* X *can take on any value in the interval* $[\underline{X}, \overline{X}]$*.*

With drastic innovations (i.e., when $m = 1$ and hence $F(g) = \frac{g}{g-1}$) Proposition 1 covers all possible cases. The analysis then confirms that generically there is no equilibrium in which the leader and outsiders simultaneously invest in R&D – a point made by Segerstrom and Zolnierek (1999). The model displays either a leapfrogging equilibrium in which only outsiders invest in R&D, or a persistent leadership equilibrium in which all the research is done by the leaders. Only in the special case $\frac{c_o}{c_\ell} = \frac{g}{g-1}$ can both leaders and outsiders invest simultaneously. In this case, the outsiders' aggregate R&D effort is $X_O = \frac{g\pi_M}{c_o} - r$, but the leader's R&D effort x_ℓ, and hence the aggregate effort X, is indeterminate. When $x_\ell = 0$ we have $X = \frac{g\pi_M}{c_o} - r$, so in equilibrium $X = \underline{X}$. As x_ℓ increases, the equilibrium interest rate increases until X_O vanishes, which occurs at $x_\ell = \overline{X}$ (since then $\frac{g\pi_M}{c_o} - r = 0$). This means that when $\frac{c_o}{c_\ell} = \frac{g}{g-1}$, the equilibrium hazard rate X can take on any value in the interval $[\underline{X}, \overline{X}]$. Intuitively, the indeterminacy of X at $\frac{c_o}{c_\ell} = \frac{g}{g-1}$ "fills" the gap between $\underline{X}$ and $\overline{X}$, ensuring that the model's equilibrium changes continuously with the exogenous parameters. (Similar remarks apply to the case $\frac{c_o}{c_\ell} = \frac{g}{g-1}$ when innovations are non drastic.)[25]

5.2 Quasi-drastic innovations: $m = 2$

With non drastic innovations, however, $F(g) < \frac{g}{g-1}$ so there exists a non-empty interval $F(g) \leq \frac{c_o}{c_\ell} < \frac{g}{g-1}$ which is not covered by Proposition 1. Hereafter we focus on that interval, in which it turns out that both outsiders and some leaders ℓ_i with $i = 1, ..., m-1$ can invest in R&D simultaneously, although x_{ℓ_m} must always vanish. To better clarify the intuitive properties of the robust equilibrium in this range, we start by analyzing separately the case of quasi-drastic innovations. Recall that with quasi-drastic innovations a two-step lead is large enough for the leader to engage in monopoly pricing, $\sqrt{\frac{1}{\alpha}} \leq \lambda < \frac{1}{\alpha}$, so that $m = 2$.

In the intermediate range, in any robust equilibrium with quasi-drastic innovations outsiders must be willing to invest in R&D, leaders who lead by one step are indifferent between investing in R&D or not, and leaders who lead by two steps do not invest. These conditions imply $X_{O_1} = \frac{\pi_1}{\tilde{V}_1} - r$ and $X_{O_2} = X_2 = \frac{\pi_M}{\tilde{V}_2} - r$, but as we have seen above in general they do not pin down x_{ℓ_1} and hence X. But now let us impose also the super-stationarity condition $X_1 = X_2 = X$. We then immediately get $X = \frac{\pi_M}{\tilde{V}_2} - r$, a decreasing relationship between X and r that, together with the Euler equation (20),

[25] In general $\underline{X}$ can be either greater or smaller than $\overline{X}$. However, in the special case $\frac{c_o}{c_\ell} = \frac{g}{g-1}$ the comparison between the leapfrogging and persistent leadership equilibria is unambiguous. To see this, consider first the case of drastic innovations (i.e., the case depicted in Figure 1). Since with $\pi_1 = \pi_M$ the graphs of the two equilibrium conditions $r = \frac{g\pi_1}{c_o} - X$ (leapfrogging) and $r = (g-1)\frac{\pi_M}{c_\ell}$ (persistent leadership) have the same intercept on the vertical axis, it is clear that $\overline{X} > \underline{X}$. With non-drastic innovations, the conclusion holds *a fortiori* since $\pi_M > \pi_1$.

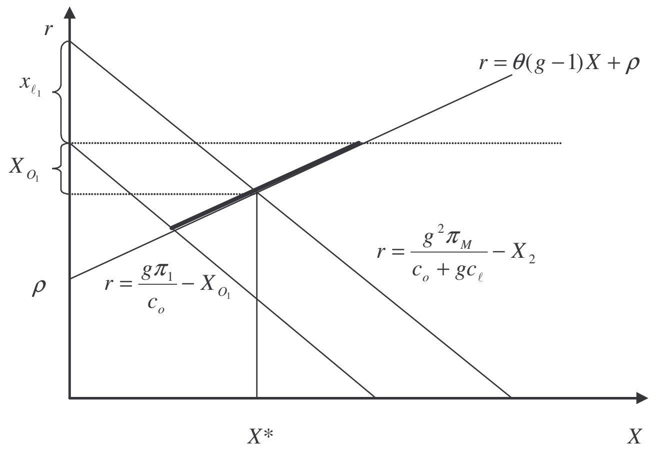

Figure 2: The super-stationary equilibrium X^* with quasi-drastic innovations. The thick line represents the set of all possible equilibria.

uniquely determines X^* and r^*.

Figure 2 illustrates. The increasing line is the Euler equation (20), and the upper decreasing line is the patent race equilibrium condition $X = \frac{\pi_M}{\tilde{V}_2} - r$. The intersection is the super-stationary equilibrium X^* and r^*. Inserting the equilibrium interest rate into the other equilibrium condition, namely $X_{O_1} = \frac{\pi_1}{\tilde{V}_1} - r$, one obtains the division of the aggregate R&D investment $X_1 = X^*$ between X_{O_1} and x_{ℓ_1}, as shown in Figure 2 (recall that the decreasing lines have slope -1). Notice that $x_{\ell_1} = \frac{\pi_M}{\tilde{V}_2} - \frac{\pi_1}{\tilde{V}_1}$.

In the case considered in Figure 2, a super-stationary equilibrium exists and hence by Lemma 5 it is robust. But now consider Figure 3. Here, if one insists that the super-stationarity condition $X_1 = X_2 = X$ must hold, the equilibrium conditions yield an interest rate that is higher than $\frac{\pi_1}{\tilde{V}_1}$, and hence too high for $X_{O_1} = \frac{\pi_1}{\tilde{V}_1} - r$ to be non negative. Therefore, in this case no super-stationary

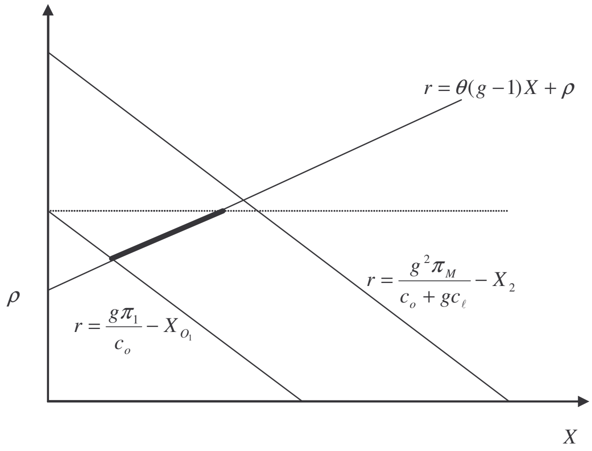

Figure 3:

equilibrium exists. In the robust equilibrium, $r = \frac{\pi_1}{\tilde{V}_1}$ so that $X_{O_1} = 0$ and $X^* = \frac{\frac{\pi_1}{\tilde{V}_1} - \rho}{\theta(g-1)}$ (from equation (20)). Since $X_2 = \frac{\pi_M}{\tilde{V}_2} - \frac{\pi_1}{\tilde{V}_1} > X^*$, one must have $x_{\ell_1} < X^*$. More precisely, x_{ℓ_1} is the solution to the equation $\kappa_1 x_{\ell_1} + \kappa_2 X_2 = X^*$, or

$$2\frac{\left(\frac{\pi_M}{\tilde{V}_2} - \frac{\pi_1}{\tilde{V}_1}\right) x_{\ell_1}}{x_{\ell_1} + \frac{\pi_M}{\tilde{V}_2} - \frac{\pi_1}{\tilde{V}_1}} = \frac{\frac{\pi_1}{\tilde{V}_1} - \rho}{\theta(g-1)}. \tag{41}$$

Summarizing this discussion:

Proposition 7 *With quasi-drastic innovations, in the intermediate range the robust equilibrium is either a super-stationary equilibrium in which* $x_{\ell_2} = 0$,

$$\begin{aligned} X &= \frac{\frac{\pi_M}{\tilde{V}_2} - \rho}{1 + \theta(g-1)} \\ x_{\ell_1} &= \frac{\pi_M}{\tilde{V}_2} - \frac{\pi_1}{\tilde{V}_1}, \end{aligned}$$

or an equilibrium in which $X_{O_1} = x_{\ell_2} = 0$,

$$\begin{aligned} X &= \frac{\frac{\pi_1}{\tilde{V}_1} - \rho}{\theta(g-1)} \\ X_{O_2} &= \frac{\pi_M}{\tilde{V}_2} - \frac{\pi_1}{\tilde{V}_1}, \end{aligned}$$

whichever results in the smaller X.

5.3 The general case

Now we turn to the general case of non drastic innovations (for drastic innovations, the interval $F(g) \leq \frac{c_o}{c_\ell} < \frac{g}{g-1}$ is empty).

5.3.1 *Super-stationary equilibria*

We start from the case in which the robust equilibrium is super-stationary. Define

$$\tilde{V}_i \equiv c_\ell \left[\frac{1}{g-1} - \left(\frac{g}{g-1} - \frac{c_o}{c_\ell} \right) g^{-i} \right]. \tag{42}$$

Notice that $\tilde{V}_i$ is increasing in i when $\dfrac{c_o}{c_\ell} < \dfrac{g}{g-1}$. Since π_i also increases with i, however, the ratio $\dfrac{\pi_i}{\tilde{V}_i}$ may either increase or decrease. In fact, it can be shown that $\dfrac{\pi_i}{\tilde{V}_i}$ first increases and then decreases with i.[26] Define i^* as the value of i that maximizes $\dfrac{\pi_i}{\tilde{V}_i}$, i.e.,

$$i^* = \arg \max_{i=1,2,\ldots,m} \left[\frac{\pi_i}{\tilde{V}_i} \right] \tag{43}$$

[26] The derivative of the ratio $\dfrac{\pi_i}{\tilde{V}_i}$ with respect to i has the same sign as

$$H = (1-\alpha) \left[\frac{c_o}{c_\ell}(g-1) - g \right] + g^i - \alpha g^{\frac{i}{\alpha}}.$$

When $i = 0$, H reduces to $(1-\alpha)\dfrac{c_o}{c_\ell}(g-1)$ and hence is positive. Thus, the ratio $\dfrac{\pi_i}{\tilde{V}_i}$ initially increases with i. But $\dfrac{dH}{di} = \left(g^i - g^{\frac{i}{\alpha}} \right) \log g < 0$, implying that $\dfrac{\pi_i}{\tilde{V}_i}$ is quasi-concave.

It can be easily checked that $i^* \geq 2$ whenever $m \geq 2$ and the condition $\frac{c_o}{c_\ell} \geq F(g)$ holds.

Proposition 8 *With non-drastic innovations and* $F(g) \leq \frac{c_o}{c_\ell} < \frac{g}{g-1}$, *if the robust equilibrium is super-stationary outsiders and all i-step leaders* ℓ_i *with* $i \leq i^* - 1$ *invest in R&D simultaneously, whereas* ℓ_{i^*} *does not invest (so no leader ever leads by more than* i^* *steps). The active leaders' equilibrium R&D efforts are*

$$x_{\ell_i} = \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - \frac{\pi_i}{\tilde{V}_i} \text{ for all } i \leq i^*, \tag{44}$$

and the equilibrium aggregate R&D effort is:

$$X = \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - r \geq x_{\ell_1}. \tag{45}$$

Several remarks are in order. First and foremost, the robust equilibrium described in Proposition 3 produces leadership cycles in which an incumbent can innovate several successive times, gradually increasing the magnitude of his technological lead before being replaced by a new entrant. Notice that since $\frac{\pi_i}{V_i}$ is increasing for $i < i^*$, equation (44) implies that a leader's R&D effort decreases with the size of his lead, i.e., $x_{\ell_i} > x_{\ell_{i+1}}$. The intuitive explanation is that initially the incumbent is eager to enlarge his lead and does much of the research. However, if he is lucky enough to innovate repeatedly, after each successive innovation his profits increase, so Arrow's replacement effect becomes stronger and stronger. Hence, the leader's incentive to invest in R&D becomes weaker and weaker, until he is eventually overtaken with probability one after i^* successive innovations, and a new cycle starts.

The fact that R&D investment by leaders decreases with their size and then vanishes has important implications for the equilibrium firm size distribution. Indeed, it follows immediately from (14) and (15) that $x_{\ell_i} > x_{\ell_{i+1}}$ and $X_i = X$ imply

$$\kappa_2 > \kappa_3 > ... > \kappa_{i^*} > \kappa_{i^*+1} = 0 \tag{46}$$

This means that the firm size distribution is skewed. The modal firm size is $i = 1$ or $i = 2$, and bigger firms are less and less frequent. This seems consistent with the empirical evidence, once one takes into account that in our model only incumbent firms are active at any point in time, implying that the model's predictions should apply only to large firms.

Moreover, Proposition 3 implies that an incumbent firm's expected rate of,

$$\frac{x_{\ell_i} g V_{i+1} - V_i}{V_i},$$

decreases with the size of the firm, V_i. This conclusion is entailed by the fact that both $\frac{V_{i+1}}{V_i}$ and x_{ℓ_i} decrease with i. It means that Gibrat's law does not hold in our model, which seems consistent with the empirical evidence for large firms (Hall, 1987; Evans, 1987).

Another noteworthy property of leadership cycles is that in innovative industries, the quality-adjusted price stays constant after each successive innovation by the leader, but jumps down when an outsider innovates. This implies that the leader alone benefits from technical progress, reaping higher and higher profits, as long as he continues to innovate. But when he is replaced by a new entrant, the cumulated benefits from all his past innovations eventually accrue to consumers. In our model with a continuum of industries this effect is smoothed out in the aggregate. However, this dynamics of the price of intermediate goods can have important macroeconomic consequences, since it affects the average price of intermediate goods, $\bar{p}$, and hence national income (for any given technology level). This implies, for instance, that policies that affect the speed with which incumbents are replaced can influence national income.

Finally, we briefly consider the comparative statics properties of the model, focusing on the cutoff value i^*. It can be easily shown that i^* increases with $\dfrac{c_o}{c_\ell}$ and tends to m as $\dfrac{c_o}{c_\ell}$ approaches $\dfrac{g}{g-1}$. One can also calculate explicitly the critical thresholds for which $i^* = j$; these are:

...

5.3.2 *Non super-stationary equilibria*

When a super-stationary equilibrium does not exist, we know that only leaders must invest in R&D in some periods. More precisely, we have:

Proposition 9 *With non-drastic innovations and $F(g) \leq \frac{c_o}{c_\ell} < \frac{g}{g-1}$, if a super-stationary equilibrium does not exist, in the robust equilibrium there exists $i^{**} \leq i^*$ such that only leaders invest in R&D for $i = 1, ..., i^{**} - 1$, both leaders and outsiders invest for $i = i^{**}, ..., i^* - 1$, and only outsiders invest for $i = i^*$ (so no leader ever leads by more than i^* steps).*

6 Welfare analysis

[to be added]

7 Conclusion

[to be added]

8 Appendix A

The proofs of Lemmas 1-4 and Propositions 1, 3 and 4 follows.

Proof of Lemma 1. Sufficiency is obvious: condition (33) is in fact a restatement of the free entry condition (27) with the associated complementary slackness condition, whereas it is immediate to check that conditions (31) and (32) guarantee that the Bellman equations are satisfied.

Hence, we focus on necessity. Notice that for all $i = 1, 2, ..., m-1$ the maximand in the Bellman equation can be rewritten as

$$\pi_i - (X_{O_i} + x_{\ell_i}) V_i + g x_{\ell_i} V_{i+1} - c_\ell x_{\ell_i},$$

and hence is linear in x_{ℓ_i} for any given X_{Oi}. As a consequence, the following inequality must hold

$$gV_{i+1} - V_i - c_\ell \leq 0,$$

for otherwise V_i is unbounded, which is inconsistent with equilibrium. Moreover, x_{ℓ_i} can be positive only if condition $gV_{i+1} - V_i - c_\ell \leq 0$ holds as an equality, whence the complementary slackness conditions

$$x_{\ell_i} (gV_{i+1} - V_i - c_\ell) = 0$$

follow. For $i = m$, the analogous condition is

$$(g-1)V_m - c_\ell \leq 0,$$

with the associated complementary slackness condition $x_{\ell_m} [(g-1)V_m - c_\ell] = 0$.

From the complementary slackness conditions it follows immediately that the system of Bellman equations (26) reduces to

$$\begin{aligned} rV_1 &= \pi_1 - X_{O_1}V_1 \\ rV_2 &= \pi_2 - X_{O_2}V_2 \\ &\dots \\ rV_m &= \pi_M - X_{O_m}V_m. \end{aligned}$$

These equations then can be solved to get (34). ■

Proof of Lemma 2. Suppose to the contrary that outsiders invest in R&D, i.e., $X_O > 0$. By (33), this requires that

$$V_1 = \frac{c_o}{g}.$$

Clearly, in equilibrium we must have $V_2 \geq V_1$, since a firm leading by two steps can always mimic a firm leading by one step only. This condition implies

$$\begin{aligned} gV_2 - V_1 - c_\ell &\geq (g-1)V_1 - c_\ell \\ &= \frac{g-1}{g}c_o - c_\ell. \end{aligned}$$

It follows that $\frac{c_o}{c_\ell} > \frac{g}{g-1}$ implies that $gV_2 - V_1 - c_\ell > 0$. But this violates (31) and hence cannot happen in equilibrium.■

Proof of Lemma 3. From (32) one sees that $x_{\ell_m} > 0$ implies $V_m = \frac{c_\ell}{g-1}$. Similarly, under the assumption that $x_{\ell_i} > 0$ for all $i = 1, 2, ...m$ one can solve the complementary slackness conditions (31) recursively, obtaining $V_1 = V_2 = ... = V_m = \frac{c_\ell}{g-1}$. On the other hand, if $\sum_{i=1}^{m} X_{O_i} > 0$ one must have $V_1 = \frac{c_o}{g}$.

But equations $V_1 = \frac{c_\ell}{g-1}$ and $V_1 = \frac{c_o}{g}$ can simultaneously hold only if $\frac{c_o}{c_\ell} = \frac{g}{g-1}$.■

Proof of Lemma 4. The proof is by contradiction. If outsiders do not invest in R&D and there is positive growth, we must have $x_{\ell_i} > 0$ for all $i = 1, 2, ...m$. Proceeding like in the proof of Lemma 3, we then get $V_1 = V_2 = ... = V_m = \frac{c_\ell}{g-1}$. But condition (33) requires that $V_1 \leq \frac{c_o}{g}$, implying $\frac{c_\ell}{g-1} \leq \frac{c_o}{g}$ or $\frac{c_o}{c_\ell} \geq \frac{g}{g-1}$, contradicting the assumption $\frac{c_o}{c_\ell} < \frac{g}{g-1}$. This contradiction establishes the Lemma.■

Proof of Proposition 1. Part (i) follows directly from Lemma 2.

To prove part (ii), notice that Lemma 4 implies that when $\frac{c_o}{c_\ell} < \frac{g}{g-1}$ outsiders must do some research, so the condition $V_1 = \tilde{V}_1$ must hold. Hence, there are only two possible cases: either a leapfrogging equilibrium where $X_O > 0$ and $x_{\ell_1} = 0$, or an equilibrium in which at least ℓ_1 invests in R&D along with outsiders, i.e., $X_O > 0$ and $x_{\ell_1} > 0$. Consider the putative leapfrogging equilibrium. Since $x_{\ell_1} = 0$, in such an equilibrium one would have

$$V_1 = \frac{\pi_1}{r + X}.$$

Condition $gV_2 - V_1 - c_\ell \leq 0$ then becomes

$$g\frac{\min[\pi_2, \pi_M]}{r + X} - \frac{\pi_1}{r + X} - c_\ell \leq 0$$

or, using the fact that $r + X = \frac{\pi_1}{V_1} = \frac{g\pi_1}{c_o}$

$$\frac{c_o \min[\pi_2, \pi_M]}{\pi_1} - \frac{c_o}{g} - c_\ell \leq 0.$$

This condition holds as a strict inequality (and hence the putative leapfrogging equilibrium is, indeed, the unique equilibrium) iff $\frac{c_o}{c_\ell} < F(g)$. If this inequality is reversed, ℓ_1 can invest in R&D in equilibrium.

Finally consider part (iii). [...]■

Proof of Proposition 3. To prove the proposition, we show that the list of variables

$$
\begin{aligned}
X &= \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - r, \\
x_{\ell_i} &= \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - \frac{\pi_i}{\tilde{V}_i} \text{ for } i = 1, ..., i^* \text{ and } x_{\ell_i} = \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - \frac{(g-1)\pi_i}{c_\ell} \text{ for } i = i^* + 1, ..., m \\
V_i &= \tilde{V}_i \text{ for } i = 1, ..., i^* \text{ and } V_i = \frac{c_\ell}{(g-1)} \text{for } i = i^* + 1, ..., m
\end{aligned}
$$

where i^* has been defined in the text, satisfies conditions (31)-(34). The result then follows from Lemma 1. Notice that the value of x_{ℓ_i} for $i = i^* + 1, ..., m$ is in fact irrelevant, since $x_{\ell_{i^*}} = 0$ implies that no leader will even gain more than i^* steps of advantage. However, one needs to show that the equilibrium values that can be observed "on the equilibrium path" can be supported by appropriate choice of "off-equilibrium-path" variables.

Notice first of all that $x_{\ell_i} \geq 0$ for all $i = 1, ..., m$. For $i = 1, ..., i^*$ this follows immediately from the definition of i^*, while for $i = i^* + 1, ..., m$ the inequality follows from the observation that $\tilde{V}_i \geq \frac{c_\ell}{(g-1)}$ and again the definition of i^*. To check that inequality $\tilde{V}_i \geq \frac{c_\ell}{(g-1)}$ indeed holds, notice that it can be re-written as

$$
c_\ell \left[\frac{1}{g-1} - \left(\frac{g}{g-1} - \frac{c_o}{c_\ell} \right) g^{-i^*} \right] \geq \frac{c_\ell}{(g-1)}
$$

or

$$c_\ell \left(\frac{g}{g-1} - \frac{c_o}{c_\ell} \right) g^{-i^*} \leq 0$$

which is obviously true when $\dfrac{c_o}{c_\ell} < \dfrac{g}{g-1}$.

It is immediate to confirm that (34) and (33) hold. It is also immediate to check that (31) holds for $i = 1, ..., i^* - 1$. Thus, it remains to show that (31) holds for $i = i^*, ..., m-1$ and that (32) holds. We distinguish between two cases.

If $i^* = m$, we must just show that $(g-1)\tilde{V}_m - c_\ell \leq 0$, that is

$$1 - \left[g - (g-1) \frac{c_o}{c_\ell} \right] g^{-m} \leq 1$$

which is obviously true when $\dfrac{c_o}{c_\ell} < \dfrac{g}{g-1}$.

If $i^* < m$, it is clear that $(g-1)V_m - c_\ell \leq 0$ and $gV_{i+1} - V_i - c_\ell \leq 0$ hold as an equality for $i > i^*$, so we must just show that $gV_{i^*+1} - V_{i^*} - c_\ell \leq 0$ holds. This inequality can be re-written as

$$\frac{gc_\ell}{(g-1)} - c_\ell \left[\frac{1}{g-1} - \left(\frac{g}{g-1} - \frac{c_o}{c_\ell} \right) g^{-i^*} \right] - c_\ell \leq 0$$

which immediately simplifies to

$$c_\ell \left(\frac{g}{g-1} - \frac{c_o}{c_\ell} \right) g^{-i^*} \leq 0$$

which is obviously true when $\dfrac{c_o}{c_\ell} < \dfrac{g}{g-1}$.

Next we show uniqueness, in the sense that in all equilibria we must have $X = \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - r$ and $x_{\ell_i} = \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - \frac{\pi_i}{\tilde{V}_i}$ for $i = 1, ..., i^*$. We know from Lemma 4 that outsiders must do some research, so the condition $V_1 = \tilde{V}_1$ must hold. We also know from Lemma 3 that in equilibrium $x_{\ell_m} = 0$. Denote by i^* the smallest integer such that $x_{\ell_{i^*}} = 0$. Since $V_1 = \tilde{V}_1$ and $x_{\ell_i} > 0$ for $i = 1, ..., i^* - 1$, one can solve the complementary slackness conditions (31) iteratively, obtaining

$$\begin{aligned} V_i &= \frac{c_o + gc_\ell + ... + g^{i-1}c_\ell}{g^i} \\ &= \frac{c_o + \dfrac{g(g^{i-1}-1)}{g-1}c_\ell}{g^i} \\ &= c_\ell\left[\frac{1}{g-1} - \left(\frac{g}{g-1} - \frac{c_o}{c_\ell}\right)g^{-i}\right] = \tilde{V}_i \end{aligned}$$

In addition, we must have $gV_{i^*} - V_{i^*-1} - c_\ell = 0$, $V_{i^*} = \dfrac{\pi_{i^*}}{r+X}$, and $V_{i^*-1} = \tilde{V}_{i^*-1}$. Combining these conditions we get $V_{i^*} = \tilde{V}_{i^*}$ and:

$$r + X = \frac{\pi_{i^*}}{\tilde{V}_{i^*}}.$$

On the other hand, the conditions $V_i = \dfrac{\pi_i}{r + X_{O_i}} = \tilde{V}_i$ give us

$$r + X_{O_i} = \frac{\pi_i}{\tilde{V}_i}$$

It follows

$$\begin{aligned} x_{\ell_i} &= X - X_{O_i} \\ &= \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - \frac{\pi_i}{\tilde{V}_i} \end{aligned}$$

as was claimed. ■

9 Appendix B

Here we develop the model with decreasing returns to scale to R&D and prove Lemma 5.

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